

Nonlinear modeling of batch digester discharge dynamics with rheology-driven hydraulic transport and drainability coupling

JOSÉ M. CAMPOS-SALAZAR

ABSTRACT: Batch digester blowdown constitutes a strongly nonlinear transient transport process governed by the coupled evolution of pulp consistency, slurry density, hydraulic resistance, liquid inventory, and discharge line dynamics. A control-oriented nonlinear formulation is developed by coupling dry fiber and free liquor mass balances with phase volume reconstruction, consistency-dependent hydraulic resistance, channeling and drainability effects, and a power law head flow relationship derived within a lumped hydraulic inertance framework. A fixed-gain PI controller obtained from local linearization is employed as the conventional benchmark, whereas an integral sliding mode controller (SMC) explicitly compensates the nonlinear time-varying hydraulic structure and bounded disturbances.

Both strategies are evaluated under identical actuator constraints and sequential perturbations involving opposing head, hydraulic resistance, channeling, drainability, and dilution. Numerical results demonstrate lower root-mean-square error, integral absolute error, and peak flow-tracking error for the SMC, with rapid recovery toward the prescribed sliding boundary layer and improved robustness to variations in the nonlinear flow exponent n . Cumulative electrical energy and batch-specific specific energy consumption remain comparatively similar, indicating that the principal SMC benefit is enhanced hydraulic regulation rather than substantial batch energy reduction. Specific energy consumption is an established metric for energetic assessment of slurry transport. Model-based hydraulic demand, electrical power, and sliding manifold surfaces further characterize the nonlinear operating domain. The resulting framework provides a computational proof-of-concept for modeling and control of transient batch digester discharge systems.

Application: The proposed model provides a simulation-based framework for evaluating batch digester discharge hydraulics and flow control performance under time-varying slurry conditions. It can support future plant-specific controller tuning, disturbance analysis, and experimental validation.

The kraft pulping process involves the simultaneous interaction of chemical reactions, heat transfer, multiphase flow, nonlinear rheology, and hydraulic transport. Among its operating stages, the discharge (blowdown) of a batch digester is one of the most critical transient operations since it governs the evacuation of concentrated pulp slurry and black liquor through a highly nonlinear hydraulic system. Its dynamic behavior directly influences production continuity, downstream washing efficiency, equipment loading, energy consumption, and overall plant productivity [1-8].

Unlike conventional hydraulic systems, concentrated pulp suspensions exhibit yield stress, shear-thinning behavior, evolving permeability, and consistency-dependent hydraulic resistance [9-13]. During blowdown, these properties continuously evolve as the fiber network restructures and dilution progresses, producing strongly nonlinear and time-varying discharge dynamics. Additional phenomena,

including channeling, drainability degradation, slurry density variation, and liquor redistribution, further modify the effective hydraulic resistance throughout the discharge process [6,10,14-16].

Several studies have investigated pulp-processing operations, including filtration, dewatering, slurry transport, and continuous digesters [3,4,17-20]. Although recent nonlinear models incorporating rheological effects have improved the prediction of pulp transport behavior [8,13,18,21], most existing approaches remain focused on steady-state conditions or simplified hydraulic representations. Consequently, the coupled effects of rheology, hydraulic resistance, slurry consistency, channeling, and drainability during transient batch digester discharge remain insufficiently represented [8,12,18,22].

The associated control problem is equally challenging. Conventional proportional-integral/proportional-integral-derivative (PI/PID) controllers remain the industrial stan-

dard because of their simplicity; however, they are typically tuned around nominal operating conditions and often experience performance degradation when confronted with hard nonlinearities, large parameter variations, and hydraulic disturbances [23–26]. In contrast, sliding mode control (SMC) has demonstrated excellent robustness against nonlinear dynamics, uncertain parameters, and external disturbances in a wide range of industrial applications [27–31]. Nevertheless, its application to batch digester discharge, coupled with a comprehensive nonlinear hydraulic-rheological model, has received limited attention.

This work addresses this gap by developing a unified nonlinear framework for batch digester discharge that combines coupled mass balances, nonlinear hydraulic transport, consistency-dependent rheology, slurry density evolution, channeling, drainability, dilution effects, and time-varying hydraulic resistance. Based on this model, a conventional fixed-gain PI controller is implemented as the industrial benchmark, while a robust SMC is proposed for discharge flow regulation under nonlinear operating conditions.

To provide a comprehensive assessment, the study also introduces hydraulic performance indicators, efficiency maps, and nonlinear operating surfaces, including sliding manifold representations, allowing detailed evaluation of tracking performance, hydraulic effort, and robustness throughout the complete blowdown process.

The principal contributions of this work are:

1. Development of a comprehensive nonlinear hydraulic-rheological model for batch digester discharge incorporating consistency-dependent transport, slurry density evolution, and time-varying hydraulic resistance.
2. Integration of industrial disturbances, including channeling, dilution, and evolving rheological properties.
3. Systematic comparison between a conventional PI controller and a robust SMC under identical nonlinear operating conditions.

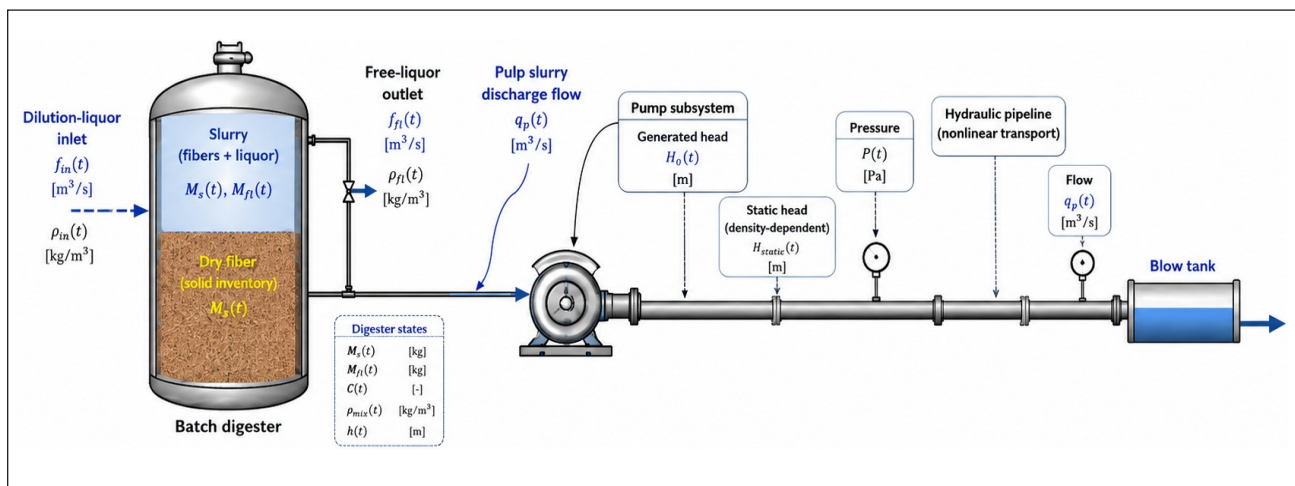
4. Introduction of hydraulic-performance indicators, efficiency maps, and sliding-manifold surfaces for comprehensive controller evaluation.
5. Demonstration that the proposed SMC significantly improves discharge-flow regulation and robustness compared with conventional PI control under nonlinear operating conditions.

The remainder of this paper covers the following topics: description of the batch digester discharge process; development of the nonlinear mathematical model; presentation of the hydraulic-performance formulation and energetic efficiency expressions; introduction to the PI and SMC; discussion of simulation results and comparative analysis; and key conclusions.

PROCESS DESCRIPTION

Figure 1 illustrates the batch digester discharge system considered in this study. This process involves the blow-down stage of a kraft batch digester, during which the cooked pulp suspension and entrapped black liquor are discharged from the pressurized vessel via the blow line towards the downstream blow tank. This operation is performed after the cooking cycle has finished in order to transfer the digester contents to the subsequent washing and separation processes [1–4].

At the beginning of the discharge operation, the digester contains a heterogeneous suspension composed of cellulose fibers, cooking liquor, dissolved lignin, and other reaction products generated during kraft pulping. As shown in Fig. 1, the suspension leaves the digester through the discharge line under the action of the available hydraulic head and is conveyed to the blow tank. During this transient process, both the amount of pulp remaining inside the digester and the free liquor inventory continuously decrease, producing simultaneous variations in slur-



1. Physical representation of the nonlinear batch digester discharge system, including the dilution liquor inlet, free liquor extraction, pulp slurry transport line, pump subsystem, and blow tank hydraulic discharge configuration.

ry consistency, suspension density, and liquid level [5-8].

The principal process variables considered throughout this study are also summarized in Fig. 1. The dry fiber mass inside the digester is represented by $M_s(t)$, whereas $M_{fl}(t)$ denotes the free liquor mass. These variables determine the instantaneous pulp consistency, $C(t)$, and the corresponding mixture density, $\rho_{mix}(t)$. The discharge flow rate is denoted by $q_p(t)$, while $q_p^*(t)$ represents its reference value. The available hydraulic head generated by the pumping system is represented by $H_0(t)$, which provides the driving force required to transport the pulp suspension through the discharge line.

The discharge system also incorporates a dilution liquor inlet, $f_{in}(t)$, and a free liquor outlet, $f_{fl}(t)$, allowing the hydraulic conditions inside the digester to evolve during blowdown. As the discharge progresses, the liquid level decreases continuously until the digester approaches its minimum operating inventory, completing the transfer of pulp suspension to the downstream process units [3,5,8].

The process considered in this study is represented as a lumped parameter hydraulic system, where the principal dynamic variables correspond to the inventories of dry fibers and free liquor inside the digester, together with the discharge flow and the hydraulic head supplied by the pumping system. These variables provide the physical basis for the nonlinear dynamic model developed in the following section.

SYSTEM MODELING

The model is developed from mass conservation, volume reconstruction, consistency-dependent resistance, and a nonlinear pressure–flow relation according Fig. 1. Similar lumped and nonlinear formulations have been employed in pulp transport and digester modeling studies [1,32-39].

The principal dynamic states are $M_s(t)$ [kg], $M_{fl}(t)$ [kg], and $q_p(t)$ [m³/s]. The $C(t)$ [-], $\rho_{mix}(t)$ [kg/m³], occupied volume $V(t)$ [m³], and equivalent slurry level $b(t)$ [m] are algebraically reconstructed from the instantaneous states.

Slurry consistency

Pulp consistency is a fundamental variable for characterizing fiber suspensions and their transport behavior. In the present lumped parameter formulation, the digester contents are represented by $M_s(t)$ and $M_{fl}(t)$. This representation provides a direct connection between the evolving material inventory and the consistency of the discharged pulp suspension. The relevance of fiber concentration to pulp suspension hydrodynamics is well established, since increasing fiber concentration promotes fiber–fiber interactions, flocculation, and changes in the resulting flow regime [7].

Accordingly, the total modeled slurry mass is defined as

$$M_{tot}(t) = M_s(t) + M_{fl}(t), \quad (1)$$

where $M_{tot}(t)$ [kg] is the total slurry mass contained in the digester.

The instantaneous pulp consistency is then expressed on a mass basis as the ratio between the $M_s(t)$ and $M_{tot}(t)$,

$$C(t) = \frac{M_s(t)}{M_{tot}(t) + \varepsilon}. \quad (2)$$

Substitution of Eq. 1 into Eq. 2 gives Eq. 3 and therefore under physically admissible conditions, $0 \leq C(t) \leq 1$. Equation 3 provides the instantaneous coupling between $M_s(t)$ and $M_{fl}(t)$ during blowdown.

$$C(t) = \frac{M_s(t)}{M_s(t) + M_{fl}(t) + \varepsilon}. \quad (3)$$

The parameter ε [kg] is a small positive numerical regularization constant introduced exclusively to prevent an ill-conditioned division as $M_s(t) + M_{fl}(t) \rightarrow 0$, and it is selected such that $\varepsilon \ll M_s(t) + M_{fl}(t)$. Consequently, any difference between their respective depletion rates produces a corresponding variation in $C(t)$. This feature becomes particularly important in the subsequent hydraulic formulation since the transport characteristics of pulp suspensions are strongly dependent on fiber concentration [7].

Phase volumes and digester level

The volumetric reconstruction of the digester inventory is obtained by considering the free liquor and fiber-containing phases separately. Consequently, the occupied volume cannot be represented solely from the free liquid inventory. Such liquid–solid descriptions are consistent with the multiphase representation commonly adopted for pulp fiber suspensions [7].

For the free liquor phase, the instantaneous density is defined as

$$\rho_{fl}(t) = \frac{M_{fl}(t)}{V_{fl}(t)}, \quad (4)$$

where $\rho_{fl}(t)$ [kg/m³] is the free liquor density and $V_{fl}(t)$ [m³] is the corresponding occupied volume. Rearranging Eq. 4 gives

$$V_{fl}(t) = \frac{M_{fl}(t)}{\rho_{fl}(t)}, \quad (5)$$

which provides the instantaneous volumetric contribution of the free liquor inventory.

The fiber-containing phase requires an additional correction since pulp fibers retain liquid within and around their porous structure. Let $w(t)$ [-] represent the effective fraction of fiber-associated liquid within the wet fiber phase. The dry solid fraction of this phase is consequently $1-w(t)$. The dry fiber inventory can therefore be related to its effective occupied volume through

$$M_s(t) = \rho_s(t) \cdot (1 - w(t)) \cdot V_s(t), \quad (6)$$

PROCESS CONTROL

where $V_s(t)$ [m³] denotes the effective volume associated with the fiber phase and $\rho_s(t)$ [kg/m³] is the effective dry fiber density. Solving Eq. 6 for $V_s(t)$ gives

$$V_s(t) = \frac{M_s(t)}{\rho_s(t) \cdot (1-w(t))}, \quad (7)$$

with the physically admissible condition $0 \leq w(t) < 1$. Here, $w(t)$ should be interpreted strictly as an effective volumetric swelling/packing parameter accounting for the additional volume associated with the wet fiber structure. It does not constitute a third independently conserved liquid mass inventory; the dynamic mass inventories remain $M_s(t)$ and $M_{fl}(t)$. The condition $0 \leq w(t) < 1$ guarantees a finite positive effective fiber phase volume [7].

Assuming additive effective phase volumes, the total occupied volume becomes

$$V(t) = V_{fl}(t) + V_s(t), \quad (8)$$

and substitution of Eq. 5 and Eq. 7 yields

$$V(t) = \frac{M_{fl}(t)}{\rho_{fl}(t)} + \frac{M_s(t)}{\rho_s(t) \cdot (1-w(t))}. \quad (9)$$

For the control-oriented model, the digester is represented by an equivalent cylindrical vessel of internal diameter, D_d [m]. Its effective cross-sectional area is

$$A_d = \frac{\pi}{4} \cdot D_d^2, \quad (10)$$

with A_d expressed in m². Under the lumped volume approximation, the instantaneous occupied volume is related to the equivalent slurry level according to

$$V(t) = A_d \cdot h(t), \quad (11)$$

where $h(t)$ [m] represents the equivalent slurry level. Therefore,

$$h(t) = \frac{V(t)}{A_d} \Rightarrow h(t) = \frac{1}{A_d} \cdot \left(\frac{M_{fl}(t)}{\rho_{fl}(t)} + \frac{M_s(t)}{\rho_s(t) \cdot (1-w(t))} \right). \quad (12)$$

Thus, $h(t)$ is not introduced as an independent dynamic state; rather, it is algebraically reconstructed from the instantaneous material inventories. As $M_s(t)$ and $M_{fl}(t)$ decrease during blowdown, the occupied volume and therefore the available hydrostatic column also evolve. This coupling is important since $h(t)$ subsequently contributes to the hydraulic head available for digester discharge.

Effective slurry density

The instantaneous slurry density is reconstructed from the total modeled mass divided by the corresponding occupied

volume. For the two-phase representation adopted here, the total mass is described in Eq. 1, while the total occupied volume follows Eq. 8. Accordingly, the effective mixture density is defined as

$$\rho_{\text{mix}}(t) = \frac{M_{fl}(t) + M_s(t)}{V_{fl}(t) + V_s(t)}, \quad (13)$$

where $\rho_{\text{mix}}(t)$ is expressed in kg/m³. The term effective is used since the pulp suspension is treated as a macroscopically homogeneous equivalent mixture rather than resolving the local spatial distribution of individual fibers and liquor [7].

Substituting Eq. 5 and Eq. 7 into Eq. 13 gives

$$\rho_{\text{mix}}(t) = \frac{M_{fl}(t) + M_s(t)}{\frac{M_{fl}(t)}{\rho_{fl}(t)} + \frac{M_s(t)}{\rho_s(t) \cdot (1-w(t))}}, \quad (14)$$

which explicitly couples the effective mixture density to the evolving fiber and free liquor inventories.

During the final stage of the numerical blowdown simulation, $M_s(t) \rightarrow 0$, $M_{fl}(t) \rightarrow 0$, and consequently $V(t) \rightarrow 0$. Equation 14 may therefore become numerically ill-conditioned even though the physical simulation is simply approaching an empty digester condition. To avoid division by a vanishingly small occupied volume, a positive volumetric regularization parameter, ε_v [m³], is introduced,

$$\rho_{\text{mix}}(t) = \frac{M_{fl}(t) + M_s(t)}{\frac{M_{fl}(t)}{\rho_{fl}(t)} + \frac{M_s(t)}{\rho_s(t) \cdot (1-w(t))} + \varepsilon_v}, \quad (15)$$

where $\varepsilon_v \ll V(t)$ over the physically relevant portion of the discharge process. Therefore, ε_v does not materially modify the reconstructed slurry density during normal operation and acts only as a numerical safeguard near the terminal inventory condition.

Equation 15 is particularly important for the subsequent hydraulic formulation since $\rho_{\text{mix}}(t)$ connects the evolving material inventory to both the transported fiber/liquor mass flows and the hydraulic-head calculations. The model therefore preserves the first-order influence of changing slurry composition while avoiding the computational complexity of a spatially distributed multiphase description.

Dry-fiber transport and mass balance

During the blowdown process, dry fibers are transported out of the digester as part of the moving pulp–liquor suspension. Within the lumped parameter formulation adopted here, the discharged suspension is represented by its bulk mixture density, instantaneous fiber mass fraction, and volumetric discharge flow. This averaged description neglects local fiber-concentration and velocity gradients

and therefore represents the fiber transport at the macroscopic control-volume level [7].

The instantaneous mass concentration of dry fibers in the discharged suspension is obtained from the product of the effective mixture density and the dry fiber mass fraction,

$$c_s(t) = \rho_{\text{mix}}(t) \cdot C(t) \quad (16)$$

where $c_s(t)$ [kg/m³] is the dry fiber mass concentration. The dry fiber mass flow transported through the digester outlet is then obtained from the general relationship between mass concentration and volumetric flow,

$$f_s(t) = c_s(t) \cdot q_p(t). \quad (17)$$

where $q_p(t)$ [m³/s] is the volumetric discharge flow. Substitution of Eq. 16 into Eq. 17 gives

$$f_s(t) = \rho_{\text{mix}}(t) \cdot C(t) \cdot q_p(t), \quad (18)$$

where $f_s(t)$ [kg/s] represents the dry fiber mass flow leaving the digester with the pulp suspension.

Equation 18 therefore establishes an important coupling between the inventory model and the hydraulic model: variations in $q_p(t)$ directly modify the rate at which dry fibers are removed, while variations in $C(t)$ and $\rho_{\text{mix}}(t)$ modify the amount of dry fiber transported per unit volume of discharged slurry. This coupling is particularly relevant for pulp suspensions since fiber concentration and fiber–fiber interactions strongly influence their multiphase transport behavior [7].

Applying conservation of dry fiber mass to the digester control volume gives the general accumulation equation

$$\frac{dM_s(t)}{dt} = f_s^{\text{in}}(t) - f_s^{\text{out}}(t), \quad (19)$$

where $f_s^{\text{in}}(t)$ and $f_s^{\text{out}}(t)$ [kg/s] denote the dry fiber inlet and outlet mass-flow rates, respectively.

During the blowdown stage considered in this study, no additional pulp fibers are introduced into the digester. Consequently, $f_s^{\text{in}}(t) = 0$. Furthermore, the dry fiber outlet flow corresponds to the transported fiber flow defined by Eq. 18, $f_s^{\text{out}}(t) = f_s(t)$. Equation 19 therefore reduces to

$$\frac{dM_s(t)}{dt} = -f_s(t) \Rightarrow \frac{dM_s(t)}{dt} = -\rho_{\text{mix}}(t) \cdot C(t) \cdot q_p(t). \quad (20)$$

The negative sign indicates the monotonic depletion of the dry fiber inventory during discharge whenever $q_p(t) > 0$. Thus, under the physically admissible conditions $\rho_{\text{mix}}(t) > 0$, $C(t) \rightarrow 0$, and $q_p(t) \rightarrow 0$, it follows that $dM_s(t)/dt \rightarrow 0$ during the blowdown operation.

Free liquor transport and mass balance

During blowdown, free liquor can leave the digester through two physically distinct mechanisms. First, a fraction of the liquor is conveyed with the discharged pulp suspension through the blow line; this contribution is denoted by $f_{\text{liq}}(t)$ [kg/s]. Second, liquor may be removed independently through the controlled extraction stream $f_{\text{fl}}(t)$ [m³/s]. These terms must therefore be treated separately in the free liquor inventory balance.

To account for the non-ideal transport characteristics of the fiber–liquor system, $f_{\text{liq}}(t)$ is subsequently corrected through two dimensionless mechanisms representing channeling and drainability.

Channeling is introduced to account for preferential flow paths through the heterogeneous fiber network, whereas the drainability contribution represents the reduction in liquor mobility associated with changes in fiber concentration and network permeability. These mechanisms are incorporated as phenomenological correction factors rather than as independently resolved spatial states, which preserves the lumped and control-oriented nature of the model [40,41].

Since $C(t)$ represents the dry fiber mass fraction (consistency), the complementary liquor mass fraction in the lumped two-phase representation is

$$l_f(t) = 1 - C(t), \quad (21)$$

where $l_f(t)$ [-] denotes the instantaneous liquor mass fraction. Accordingly, $0 \leq C(t) \leq 1$ and $0 \leq l_f(t) \leq 1$. The corresponding liquor mass concentration transported by the bulk slurry can then be approximated as

$$c_{\text{liq}}(t) = \rho_{\text{mix}}(t) \cdot l_f(t) \Rightarrow c_{\text{liq}}(t) = \rho_{\text{mix}}(t) \cdot (1 - C(t)), \quad (22)$$

where $c_{\text{liq}}(t)$ is expressed in kg/m³. Multiplication by the volumetric pulp discharge flow $q_p(t)$ gives the nominal liquor mass-flow rate ($f_{\text{liq}}^0(t)$),

$$f_{\text{liq}}^0(t) = c_{\text{liq}}(t) \cdot q_p(t) \Rightarrow f_{\text{liq}}^0(t) = \rho_{\text{mix}}(t) \cdot (1 - C(t)) \cdot q_p(t). \quad (23)$$

Equation 23 represents the nominal case in which the liquor fraction is transported uniformly with the bulk pulp flow. To account for deviations from this ideal transport condition, a dimensionless channeling factor is introduced directly into the model. The parameter $k_{\text{ch}}(t)$ represents the fraction of the nominal liquor transport capacity that is effectively lost since part of the flow follows preferential paths through the heterogeneous fiber network [40]. Accordingly, the effective channeling correction is defined as

$$r_{\text{ch}}(t) = 1 - k_{\text{ch}}(t), \quad (24)$$

where $k_{ch}(t)$ [-] is the channeling factor. The admissible range $0 \leq k_{ch}(t) \leq 1$ implies $0 \leq r_{ch}(t) \leq 1$. Thus, $k_{ch}(t) = 0$ represents the idealized absence of channeling, whereas increasing $k_{ch}(t)$ progressively reduces the effective amount of liquor transported with the bulk suspension.

A second correction is introduced to represent the evolution of the effective drainability of the pulp fiber network. In the present lumped formulation, drainability is associated with the capability of free liquor to migrate through and leave the deformable fiber structure during blowdown. As the local fiber concentration increases, the interconnected network becomes progressively less permeable and offers greater resistance to liquid transport [41]. A dimensionless drainability factor is therefore defined as

$$r_d(t) = 1 - y_K(t) \cdot C(t), \quad (25)$$

where $y_K(t)$ [-] is the drainability sensitivity coefficient. For physical consistency, the parameters are constrained such that $0 \leq r_d(t) \leq 1$ [41].

The actual free liquor mass flow transported with the pulp suspension is consequently expressed as

$$\begin{aligned} \dot{m}_{liq}(t) &= \dot{m}_{liq}^0(t) \cdot r_{ch}(t) \cdot r_d(t) \Rightarrow \dot{m}_{liq}(t) = \\ & (1 - k_{ch}(t)) \cdot (1 - y_K(t) \cdot C(t)) \cdot \rho_{mix}(t) \cdot (1 - C(t)) \cdot q_p(t). \end{aligned} \quad (26)$$

Equation 26 constitutes a phenomenological phase participation closure. Consequently, the product $r_{ch}(t) \cdot r_d(t)$ modifies the fraction of nominal liquor transport participating in the bulk discharge and should not be interpreted as an exact homogeneous no-slip phase flux relation.

The dynamic free liquor inventory follows from the macroscopic conservation equation

$$\frac{dM_{fl}(t)}{dt} = \dot{m}_{fl}^i(t) - \dot{m}_{fl}^{ext}(t) - \dot{m}_{liq}(t), \quad (27)$$

where all three terms on the right-hand side are mass-flow rates [kg/s]. The dilution-liquor inflow is obtained from its volumetric flow as

$$\dot{m}_{fl}^i(t) = \rho_{in} \cdot \dot{V}_{in}(t), \quad (28)$$

where $\dot{V}_{in}(t)$ [m³/s] is the dilution-liquor volumetric flow and $\rho_{in}(t)$ [kg/m³] is its density. Similarly, the independently extracted liquor mass flow is

$$\dot{m}_{fl}^{ext}(t) = \rho_{fl}(t) \cdot \dot{V}_{fl}(t), \quad (29)$$

where ρ_{fl} [kg/m³] is the free liquor density and $\dot{V}_{fl}(t)$ [m³/s] is the controlled free liquor extraction flow.

Substituting Eq. 28 and Eq. 29 into Eq. 27 gives the final free liquor state equation,

$$\frac{dM_{fl}(t)}{dt} = \rho_{in} \cdot \dot{V}_{in}(t) - \rho_{fl} \cdot \dot{V}_{fl}(t) - \dot{m}_{liq}(t). \quad (30)$$

Consistency-dependent hydraulic resistance

The hydraulic transport resistance of a pulp suspension is strongly affected by fiber concentration since increasing consistency intensifies fiber-fiber interactions, floc formation, network strength, and frictional interaction with the surrounding liquid and pipe wall. In addition, it is known that pressure losses depend strongly on pulp consistency, velocity, pipe diameter, and fiber characteristics; in the plug-flow regime, consistency was reported as one of the dominant variables governing the measured pressure drop [15].

To express this dependence in dimensionless form, a normalized consistency ratio is first introduced as

$$C_r(t) = \frac{(C(t) + \varepsilon_c)}{C^*}, \quad (31)$$

where $C_r(t)$ [-] is the normalized consistency, C^* [-] is a prescribed reference consistency, and ε_c [-] is a small positive regularization parameter. In the present formulation, C^* is a constant reference value, rather than a time-dependent state, and provides the operating condition relative to which changes in hydraulic resistance are evaluated [15,42].

For $C(t) = C^*$ and assuming $\varepsilon_c \ll C^*$, it follows that $C_r(t) \approx 1$. The effective hydraulic resistance is subsequently represented through the empirical power-law relationship

$$C_n(t) = K^* \cdot C_r^{\alpha_c}(t) \Rightarrow C_n(t) = K^* \cdot \left(\frac{C(t) + \varepsilon_c}{C^*} \right)^{\alpha_c}, \quad (32)$$

where $C_n(t)$ is the effective nonlinear hydraulic-resistance coefficient, K^* is its calibrated reference value, and α_c [-] is the consistency sensitivity exponent. At the reference operating condition, $C(t) \approx C^*$, Eq. 32 reduces approximately to $C_n(t) \approx K^*$. Therefore, K^* can be interpreted as the nominal hydraulic resistance coefficient associated with the reference consistency [15,42].

The exponent α_c determines how rapidly the hydraulic resistance changes as the pulp consistency departs from its reference value. For example, $\alpha_c = 0$ would produce a consistency-independent resistance, $C_n(t) \approx K^*$, whereas $\alpha_c > 0$ produces increasing resistance as $C(t)$ increases [15,42]. The sensitivity can be seen directly by differentiating Eq. 32. Neglecting the small regularization term for interpretation,

$$\lim_{\varepsilon_c \rightarrow 0} K^* \cdot \left(\frac{C(t) + \varepsilon_c}{C^*} \right)^{\alpha_c} \approx K^* \cdot \left(\frac{C(t)}{C^*} \right)^{\alpha_c} \Rightarrow \frac{C_n(t)}{K^*} \approx \left(\frac{C(t)}{C^*} \right)^{\alpha_c}, \quad (33)$$

and then taking logarithms from Eq. 33 and solving for α_c gives

$$\alpha_c = \frac{\ln\left(\frac{C_n(t)}{K^*}\right)}{\ln\left(\frac{C(t)}{C^*}\right)}, \quad (34)$$

which provides a convenient interpretation of α_c : it is the logarithmic sensitivity of hydraulic resistance to changes in pulp consistency. Thus, larger values of α_c represent stronger hydraulic sensitivity to changes in fiber concentration.

Static hydraulic contribution

Changes in slurry composition during blowdown modify the effective mixture density and, consequently, the pressure associated with a given fluid column. In the present lumped formulation, this density-dependent contribution is incorporated through an equivalent static hydraulic head. Such coupling is relevant in slurry-transport systems since variations in solids content, density, viscosity, and elevation affect the pressure distribution and hydraulic requirements of the pipeline system [43].

For a homogeneous fluid column of effective height z_{eff} [m], the hydrostatic pressure contribution can be written as [44,45]

$$\Delta p_{\text{static}}(t) = \rho_{\text{mix}}(t) g z_{\text{eff}}, \quad (35)$$

where $\Delta p_{\text{static}}(t)$ [Pa] is the equivalent hydrostatic pressure difference, and g [m/s²] is the gravitational acceleration.

If this pressure contribution is expressed as an equivalent hydraulic head referred to a nominal density ρ^* [kg/m³], then

$$H_{\text{static}}(t) = \frac{\Delta p_{\text{static}}(t)}{g \rho^*}. \quad (36)$$

Substitution of Eq. 35 into Eq. 36 gives

$$H_{\text{static}}(t) = \frac{\rho_{\text{mix}}(t) z_{\text{eff}}}{\rho^*}. \quad (37)$$

Defining the lumped coefficient

$$K_{\text{static}} = \frac{z_{\text{eff}}}{\rho^*}, \quad (38)$$

the density-dependent contribution can be expressed in the compact control-oriented form

$$H_{\text{static}}(t) = K_{\text{static}} \rho_{\text{mix}}(t), \quad (39)$$

where $H_{\text{static}}(t)$ [m] represents the equivalent static hydraulic contribution and K_{static} is an effective density-to-head coefficient [46,47].

Equation 39 is a lumped density-dependent static-head representation. Here, z_{eff} may be calibrated for the particu-

lar digester blow line geometry. Accordingly, Eq. 38 combines the effective elevation with the selected reference-density normalization. Additional dynamic or unresolved hydraulic losses are represented separately through the disturbance-head term introduced subsequently.

Available hydraulic head

The discharge flow is driven by the net hydraulic energy available between the digester and the downstream discharge line. In the present lumped formulation, this available head results from the combined contribution of the pump-generated head and the hydrostatic level of the slurry remaining inside the digester, reduced by the equivalent static hydraulic contribution introduced previously. This type of head decomposition is consistent with dynamic slurry pipeline formulations in which pressure, elevation, density, and hydraulic losses jointly determine the instantaneous transport conditions [43].

The total driving head can therefore be written in general form as

$$H_{\text{av}}(t) = H_{\text{driving}}(t) - H_{\text{opposing}}(t). \quad (40)$$

For the present system, the driving contribution consists of the pump head and the instantaneous slurry level,

$$H_{\text{driving}}(t) = H_0(t) + H_b(t), \quad (41)$$

where $H_0(t)$ [m] is the hydraulic head supplied by the pumping system. In addition, $H_b(t) = \frac{\rho_{\text{mix}}(t)}{\rho^*} b(t)$ is the level head [m]. The opposing contribution is represented by the density-dependent static term,

$$H_{\text{opposing}}(t) = H_{\text{static}}(t). \quad (42)$$

Substitution of Eq. 41 and Eq. 42 into Eq. 40 yields

$$H_{\text{av}}(t) = H_0(t) + \frac{\rho_{\text{mix}}(t)}{\rho^*} b(t) - H_{\text{static}}(t), \quad (43)$$

where $H_{\text{av}}(t)$ [m] denotes the net hydraulic head available to drive the pulp suspension through the blow line [47].

The sign convention adopted in Eq. 43 has a direct physical interpretation. The terms $H_0(t)$ and $b(t)$ appear with positive signs since both contribute hydraulic energy to the discharge process. In contrast, $H_{\text{static}}(t)$ is subtracted since it represents an equivalent hydraulic requirement that must be overcome before the remaining head can be converted into discharge flow.

In addition, Eq. 43 also establishes an additional nonlinear coupling between the material inventories and the hydraulic subsystem. Since $b(t) = f(M_s(t), M_n(t))$, and $H_{\text{static}}(t) = g(\rho_{\text{mix}}(t))$, with $\rho_{\text{mix}}(t) = k(M_s(t), M_n(t))$, the available hydraulic head can be written more generally as $H_{\text{av}}(t)$

$= r(H_0(t), M_s(t), M_{fl}(t))$. Also, $f(\cdot)$, $g(\cdot)$, $l(\cdot)$, and $r(\cdot)$ are mathematical functions.

Therefore, even for a constant pump command $H_0(t)$, the available head can vary continuously during blowdown as the slurry inventories, level, and effective density evolve. This coupling is one of the mechanisms responsible for the time-varying nonlinear discharge behavior studied in the subsequent hydraulic flow formulation.

For a more detailed industrial model, Eq. 43 could be extended to include distributed frictional losses, fittings, downstream backpressure, and additional disturbances,

$$H_{av}(t) = H_0(t) + \frac{\rho_{mix}(t)}{\rho^*} \cdot b(t) - H_{static}(t) - H_{loss}(t) - H_{back}(t); \quad (44)$$

however, the compact form in Eq. 43 is retained here to preserve the control-oriented nature of the proof-of-concept model [47].

Nonlinear resistive head and equilibrium discharge flow

The instantaneous discharge capacity of the blow line is represented through a nonlinear relationship between the available hydraulic head and the corresponding equilibrium volumetric flow. This formulation is motivated by the behavior of shear-dependent fluids, for which the pressure drop is generally not proportional to flow rate. For a power-law fluid [48,49],

$$\tau = K_r \cdot \dot{\gamma}^n, \quad (45)$$

where τ [Pa] is the shear stress, K_r [Pa·sⁿ] is the rheological consistency index, $\dot{\gamma}$ [s⁻¹] is the shear rate, and n [-] is the flow-behavior index. Analytical pipe-flow solutions for this constitutive model yield a pressure-drop relation of the general form [48,49]

$$\Delta P(t) = K_Q \cdot Q^n(t), \quad (46)$$

where $Q(t)$ [m³/s] is the volumetric flow rate and K_Q incorporates the rheological and geometric characteristics of the conduit. In particular, Sochi [48] derived explicitly for laminar flow of a power-law fluid through circular and axisymmetric tubes that the pressure drop is proportional to the n -th power of the volumetric flow rate.

Expressing pressure difference as an equivalent hydraulic head, similar to Eq. 36,

$$H(t) = \frac{\Delta P(t)}{\rho^* \cdot g}, \quad (47)$$

allows Eq. 46 to be rewritten in the control-oriented form

$$H_{av}(t) = C_n(t) \cdot q_{p,alg}^n(t) \Rightarrow q_{p,alg}(t) = \left(\frac{H_{av}(t)}{C_n(t)} \right)^{\frac{1}{n}}, \quad (48)$$

where $q_{p,alg}(t)$ [m³/s] is the instantaneous algebraic or equilibrium discharge flow.

In the present lumped model, Eq. 48 should be interpreted as a reduced nonlinear hydraulic closure, rather than as an exact spatially distributed pipe flow solution. The detailed effects of pipe geometry, fiber network resistance, consistency, and unresolved hydraulic losses are incorporated into $C_n(t)$, while the exponent n determines the curvature of the head–flow relationship. The power-law form is nevertheless physically motivated by analytical non-Newtonian pipe flow solutions in which pressure drop scales as $Q_n(t)$ [48,49].

The physical interpretation of Eq. 48 is particularly important. For a fixed resistance $C_n(t)$, $q_{p,alg}(t) \propto H_{av}^{1/n}(t)$, whereas, for a fixed $H_{av}(t)$, $q_{p,alg}(t) \propto C_n^{-1/n}(t)$. Consequently, increasing $H_{av}(t)$ increases the discharge capacity, whereas increasing the effective hydraulic resistance decreases it. The parameter n controls the sensitivity of both relationships. For $n = 1$, Eq. 48 reduces to the linear resistance law

$$q_{p,alg}(t) = \frac{H_{av}(t)}{C_n(t)}, \quad (49)$$

whereas $n \neq 1$ produces the nonlinear behavior required to represent shear-dependent hydraulic transport.

For numerical implementation, two physically meaningful lower bounds are introduced. First,

$$H_{eff}(t) = \max(H_{av}(t), H_\epsilon), \quad (50)$$

where $H_\epsilon > 0$ [m] prevents evaluation of a fractional power at a negative or vanishing hydraulic head. Second,

$$C_{n,eff}(t) = \max(C_n(t), C_{n,min}), \quad (51)$$

where $C_{n,min} > 0$ has the same engineering units as $C_n(t)$ and avoids division by a vanishing resistance coefficient.

Therefore, considering Eq. 50 and Eq. 51, replacing them into Eq. 48 gives,

$$q_{p,alg}(t) = \left(\frac{\max(H_{av}(t), H_\epsilon)}{\max(C_n(t), C_{n,min}) + \epsilon_{C_n}} \right)^{\frac{1}{n}}, \quad (52)$$

where ϵ_{C_n} is a small positive regularization term having the same units as $C_n(t)$. Equation 52 is a numerically protected algebraic flow expression.

Finally, $q_{p,alg}(t)$ should not be confused with the actual measured discharge flow $q_p(t)$. Equation 52 represents the instantaneous hydraulic equilibrium flow that would be obtained for the current values of $H_{av}(t)$, $C_n(t)$, and n . The actual flow approaches this equilibrium according to the finite hydraulic dynamics introduced in the subsequent subsection.

Dynamic discharge flow response

The algebraic flow $q_{p,alg}(t)$ defined previously represents the instantaneous equilibrium discharge rate associated with the current $H_{av}(t)$ and nonlinear transport resistance. However, the actual discharge flow $q_p(t)$ cannot change instantaneously since the slurry contained in the blow line possesses finite inertance. Accordingly, the dynamic flow equation is obtained from a lumped one-dimensional momentum balance applied to an equivalent rigid slurry column.

The derivation assumes an effective pipe segment of constant cross-sectional area A_p [m²] and length L_{eff} [m], with the mixture density considered approximately uniform along the line over the hydraulic transient. This is consistent with the rigid column approximation commonly employed for relatively slow hydraulic transients and control-oriented simulations [50].

The volume of slurry contained in the effective discharge line is

$$V_p = A_p \cdot L_{eff}, \quad (53)$$

and its lumped mass is therefore

$$M_p(t) = \rho_{mix}(t) \cdot V_p. \quad (54)$$

For a uniform average axial velocity $v_p(t)$ [m/s],

$$q_p(t) = A_p \cdot v_p(t) \Rightarrow v_p(t) = \frac{q_p(t)}{A_p}, \quad (55)$$

and, for constant A_p ,

$$\frac{dv_p(t)}{dt} = \frac{1}{A_p} \cdot \frac{dq_p(t)}{dt}. \quad (56)$$

Applying Newton's second law to the equivalent slurry column gives

$$(\Delta p(t) - \Delta p_R(t)) \cdot A_p = M_p(t) \cdot \frac{dv_p(t)}{dt}, \quad (57)$$

where $\Delta p(t)$ [Pa] is the net driving pressure difference and $\Delta p_R(t)$ [Pa] is the resistive pressure loss.

Substitution of Eq. 54 and Eq. 56 into Eq. 57 and considering Eq. 53, yields

$$\begin{aligned} (\Delta p(t) - \Delta p_R(t)) \cdot A_p &= \frac{\rho_{mix}(t) \cdot V_p}{A_p} \cdot \frac{dq_p(t)}{dt} \Rightarrow \Delta p(t) = \\ &\frac{\rho_{mix}(t) \cdot L_{eff}}{A_p} \cdot \frac{dq_p(t)}{dt} + \Delta p_R(t), \end{aligned} \quad (58)$$

which corresponds to the classical inertance resistance structure used in lumped hydraulic models. Hydraulic inertance proportional to $\rho \cdot L/A$ and equations of the form

$I(t) \cdot dQ(t)/dt + R \cdot Q(t) = \Delta p(t)$ are well established in lumped hydraulic modeling [51]. The corresponding pressure form hydraulic inertance is therefore

$$I_p(t) = \frac{\rho_{mix}(t) \cdot L_{eff}}{A_p}. \quad (59)$$

In the present model, $\rho_{mix}(t)$ evolves with the slower inventory dynamics. Equation 58 consequently assumes that its variation during an individual hydraulic transient is sufficiently slow that the additional momentum term associated explicitly with $d\rho_{mix}(t)/dt$ can be neglected. This approximation is consistent with the intended reduced order character of the model.

Since the static head model developed previously is referenced to the nominal density ρ^* , the driving and resistive pressure differences are expressed using the same reference density head definition

$$\begin{cases} H_{av}(t) = \frac{\Delta p(t)}{\rho^* \cdot g} \\ H_R(t) = \frac{\Delta p_R(t)}{\rho^* \cdot g} \end{cases} \quad (60)$$

The use of a reference density equivalent or carrier fluid equivalent head is consistent with slurry pipeline practice, where pressure gradients and pump heads may be expressed relative to a specified carrier fluid or water column reference [47].

Substitution of Eq. 60 into Eq. 58 gives

$$\begin{aligned} \rho^* \cdot g \cdot H_{av}(t) &= I_p(t) \cdot \frac{dq_p(t)}{dt} + \rho^* \cdot g \cdot H_R(t) \Rightarrow \\ H_{av}(t) &= \frac{1}{g \rho^*} \cdot I_p(t) \cdot \frac{dq_p(t)}{dt} + H_R(t); \end{aligned} \quad (61)$$

thus, the corresponding head-domain hydraulic inertance is

$$I_H(t) = \frac{\rho_{mix}(t)}{\rho^*} \cdot \frac{L_{eff}}{g \cdot A_p}. \quad (62)$$

Analytical power-law pipe flow solutions explicitly produce pressure drop relations proportional to $Q_n(t)$, thereby providing the physical basis for the present reduced nonlinear resistance law [48]. For the blowdown direction $q_p(t) \geq 0$,

$$H_R(t) = C_n(t) \cdot q_p^n(t). \quad (63)$$

If flow reversal were to be admitted, the signed form

$$H_R(t) = C_n(t) \cdot q_p(t) \cdot |q_p(t)|^{n-1} \quad (64)$$

can preserve the direction of the resistive contribution. In the present blowdown application, however, the physically admissible operating region is restricted to $q_p(t) \geq 0$, and Eq. 63 is therefore sufficient.

Substitution of Eq. 63 into Eq. 61 yields the nonlinear lumped momentum equation,

$$H_{av}(t) = I_H(t) \cdot \frac{dq_p(t)}{dt} + C_n(t) \cdot q_p^n(t) \Rightarrow \quad (65)$$

$$\frac{dq_p(t)}{dt} = \frac{1}{I_H(t)} \cdot \left(H_{av}(t) - C_n(t) \cdot q_p^n(t) \right).$$

Equation 65 is the nonlinear dynamic discharge flow equation obtained directly from the reduced momentum balance.

HYDRAULIC AND ENERGETIC EFFICIENCY FORMULATION

The energetic performance of the batch digester discharge process is evaluated from the hydraulic power required to maintain slurry transport under the time-varying resistance, density, consistency, and hydraulic head conditions developed previously. This distinction is relevant for slurry pipelines, since transport energy and specific power consumption depend strongly on the hydraulic resistance and transported solids concentration; consequently, energetic performance should be evaluated together with the evolving transport state rather than from pump efficiency alone [52-54].

Since the hydraulic heads in the present formulation are expressed with respect to the nominal reference density ϱ^* [kg/m³], the instantaneous hydraulic power delivered by the pumping subsystem is written as

$$P_h(t) = \varrho^* \cdot g \cdot q_p(t) \cdot H_0(t), \quad (66)$$

where $P_h(t)$ [W] is the hydraulic power transferred by the pump. If the head were instead expressed directly in meters of the instantaneous slurry, ϱ^* would be replaced by $\varrho_{mix}(t)$. The conventional hydraulic power relation $P_h(t) = \varrho g Q(t) \cdot H(t)$ and the corresponding pump-efficiency definition are well established in pump performance analysis [54,55].

Introducing the pump and motor efficiencies, $\eta_p(t)$ [-] and $\eta_m(t)$ [-], respectively, the estimated electrical input power becomes

$$P_{el}(t) = \frac{P_h(t)}{\eta_p(t) \cdot \eta_m(t)} \Rightarrow P_{el}(t) = \frac{\varrho^* \cdot g \cdot q_p(t) \cdot H_0(t)}{\eta_p(t) \cdot \eta_m(t)}. \quad (67)$$

Nonlinear transport power

The nonlinear resistive head developed in the hydraulic model is related to Eq. 63. The instantaneous hydraulic power associated with overcoming this nonlinear slurry transport resistance is therefore

$$P_R(t) = \varrho^* \cdot g \cdot q_p(t) \cdot H_R(t). \quad (68)$$

Substitution of Eq. 63 gives

$$P_R(t) = \varrho^* \cdot g \cdot C_n(t) \cdot q_p^{n+1}(t). \quad (69)$$

Equation 69 explicitly demonstrates the nonlinear energetic coupling of the process. Even for constant $C_n(t)$, $P_R(t) \propto q_p^{n+1}(t)$, whereas an increase in consistency modifies $C_n(t)$ and therefore directly changes the hydraulic power required to sustain a specified discharge flow. Experimental and analytical slurry pipeline studies similarly show that pressure loss and specific energy consumption (SEC) depend strongly on solids concentration and operating flow conditions [52-55].

Model-based hydraulic demand partition

To distinguish the energy required for effective slurry conveyance from other hydraulic requirements, define the auxiliary head as

$$H_{aux}(t) = H_{static}(t) + H_{dist}(t), \quad (70)$$

where $H_{dist}(t)$ [m] represents unresolved downstream, back-pressure, or additional hydraulic loss contributions when included in the simulation. The corresponding total hydraulic requirement is

$$H_{req}(t) = H_R(t) + H_{aux}(t). \quad (71)$$

The hydraulic transport-efficiency index is then defined as the fraction of the hydraulic requirement associated directly with slurry conveyance,

$$\chi_R(t) = \frac{H_R(t)}{H_{req}(t)} \Rightarrow \chi_R(t) = \frac{C_n(t) \cdot q_p^n(t)}{C_n(t) \cdot q_p^n(t) + H_{aux}(t)}, \quad (72)$$

where $\chi_R(t)$ [-] is the resistive head fraction, representing the fraction of the modeled hydraulic requirement associated with nonlinear slurry transport resistance.

The corresponding overall transport indicator, including nominal pump-motor conversion losses, is

$$\eta_{tr}(t) = \eta_p(t) \cdot \eta_m(t) \cdot \chi_R(t). \quad (73)$$

For positive head contributions, $0 \leq \chi_R(t) \leq 1$ and $0 \leq \eta_{tr}(t) \leq 1$. Importantly, $\eta_{tr}(t)$ should be described as a model-based transport efficiency indicator, not as a universal thermodynamic or exergy efficiency. It quantifies how the hydraulic demand is distributed between nonlinear slurry conveyance and the additional static/disturbance requirements represented by the reduced order model [47,54,56,57].

Electrical energy and specific energy consumption

For controller comparison, instantaneous efficiency alone is insufficient since two controllers may achieve similar tracking while requiring different hydraulic actuation. The cumulative electrical energy consumed during a blowdown operation is therefore

$$E_{el}(t) = \int_0^t P_{el}(\beta) \cdot d\beta \quad (74)$$

with $E_{el}(t)$ expressed in J or, after conversion, kWh.

A particularly useful process-level metric is the electrical energy consumed per unit dry fiber discharged, defined as

$$\Delta M_s = M_s(0) - M_s(t_f), \quad (75)$$

where ΔM_s [kg] is the total dry fiber mass discharged during the considered blowdown interval; $M_s(0)$ [kg] is the initial dry fiber mass contained in the digester at the beginning of the discharge process, $t = 0$; $M_s(t_f)$ [kg] is the remaining dry fiber mass in the digester at the final simulation or discharge time t_f ; and t_f [s] is the final time of the analyzed blowdown process.

The SEC becomes

$$SEC_{s,batch} = \frac{E_{el}(t_f)}{M_s(0) - M_s(t_f)}, \quad (76)$$

with units J/kg of dry fiber, or conventionally kWh/t after unit conversion. This batch-specific SEC is analogous to conventional slurry transport SEC but is normalized here by the total dry fiber mass discharged during the finite blowdown interval. Specific energy consumption is widely employed for energetic comparison of slurry pipeline operating conditions, since it directly relates pumping energy to the amount of transported solids [52-54,56].

Hydraulic-demand and power surfaces

The nonlinear formulation additionally permits construction of the efficiency maps used in the simulation analysis. For a prescribed auxiliary head $H_{aux}^*(t)$, Eq. 73 becomes [47,58,59]

$$\eta_{tr}\left(q_p(t), C_n(t)\right) = \eta_p \cdot \eta_m \cdot \frac{C_n(t) \cdot q_p^n(t)}{C_n(t) \cdot q_p^n(t) + H_{aux}^*}. \quad (77)$$

Equation 77 defines the efficiency surface in the $(q_p(t), C_n(t))$ operating plane and provides a compact visualization of how hydraulic performance changes with discharge rate and consistency-dependent resistance [47,58,59].

Likewise, at hydraulic equilibrium, the pump head required to sustain a specified operating point follows from the head balance as

$$H_{0,req}(t) = C_n(t) \cdot q_p^n(t) + H_{aux}(t) - H_h(t). \quad (78)$$

Consistently with the reference-density convention (see Eq. 41), the resulting electrical power surface is therefore

$$P_{el}\left(q_p(t), C_n(t)\right) = \frac{\varrho^* \cdot g \cdot q_p(t)}{\eta_p \cdot \eta_m} \cdot \left(C_n(t) \cdot q_p^n(t) + H_{aux} - H_h\right). \quad (79)$$

Only physically feasible operating points satisfying $0 \leq H_{0,req}(t) \leq H_{0,max}$ are retained in the efficiency and power maps. These surfaces are model-based operating maps, not experimentally-identified pump efficiency maps; their purpose is to visualize the energetic consequences of the nonlinear hydraulic model and to compare the PI-controlled and SMC-controlled trajectories across the same operating domain.

CONTROL SYSTEM DESIGN

The control goal is to regulate the blow line discharge flow $q_p(t)$ around a prescribed reference $q_p^*(t)$, despite the time-varying slurry consistency, density, hydraulic resistance, digester level, and unmeasured hydraulic disturbances developed in the preceding model. A conventional fixed-gain PI controller is retained as the linear benchmark, while a nonlinear SMC, as shown in **Fig. 2**, is designed explicitly from the hydraulic dynamics [60].

Control-oriented hydraulic model

Using the available-head definition developed previously in Eq. 43,

$$H_{av}(t) = H_0(t) + H_h(t) - H_{static}(t) - d_H(t), \quad (80)$$

where $H_0(t)$ [m], the hydraulic head supplied by the pumping system, also plays the role of manipulated variable. In addition, $d_H(t)$ [m] represents unmodeled or external opposing head disturbances. Finally, Eq. 65 becomes [61]:

$$\frac{dq_p(t)}{dt} = \frac{1}{I_H(t)} \cdot \left(H_0(t) + H_h(t) - H_{static}(t) - C_n(t) \cdot q_p^n(t) - d_H(t)\right). \quad (81)$$

Here,

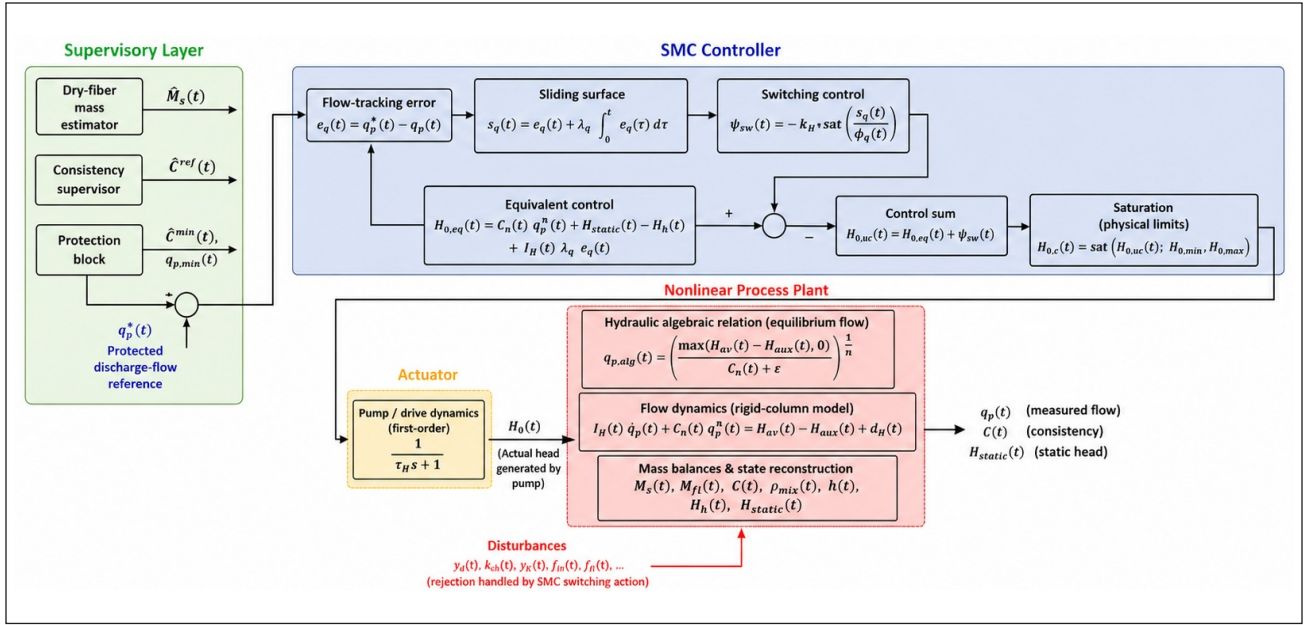
$$I_H(t) = \frac{\varrho_{max}(t)}{\varrho^*} \cdot \frac{L_{eff}}{g \cdot A_p} > 0 \quad (82)$$

is the head-domain hydraulic inertance introduced in Eq. 62. Equation 81 makes explicit the control-relevant nonlinearities: $I_H(t)$ varies with slurry density; $C_n(t)$ varies with consistency; and $H_h(t)$ and $H_{static}(t)$ vary as the digester inventory evolves.

For both controllers, the flow-tracking error is defined consistently as

$$e_q(t) = q_p^*(t) - q_p(t), \quad (83)$$

where $q_p^*(t)$ [m³/s] is the desired blow line discharge flow.



2. Proposed sliding mode controller (SMC) architecture for nonlinear discharge flow regulation in the batch digester system, including the supervisory layer, hydraulic actuator dynamics, nonlinear process model, and disturbance rejection structure.

Fixed-gain PI

A conventional PI controller is adopted as a transparent baseline for comparison with the nonlinear SMC. The PI controller is designed from a single local linearization of the nonlinear hydraulic dynamics in Eq. 81, and its gains remain unchanged throughout the complete blowdown. Thus, no nonlinear inverse compensation, gain scheduling, adaptation, or direct use of the time-varying resistance $C_n(t)$ is introduced into the benchmark. This approach is consistent with conventional model-based PI design, in which a reduced low-order representation of the process is used to determine fixed controller parameters [62].

To develop the linearization of the model, the steady-state variables are presented, consolidated into the nominal-point vector, $\bar{\mathbf{P}} = (\bar{q}_p \ \bar{C}_n \ \bar{I}_H \ \bar{H}_h \ \bar{H}_{static} \ \bar{H}_0)$ where symbolically $\bar{\mathbf{P}} \in \{\mathbb{R}^6\}$ [63]. At the nominal operating condition, the discharge flow is assumed to be at hydraulic equilibrium, $\left. \frac{dq_p(t)}{dt} \right|_{\bar{\mathbf{P}}} = 0$, and, in the absence of the imposed disturbance $d_H(t)$, Eq. 81 gives

$$0 = \frac{1}{\bar{I}_H} \cdot (\bar{H}_0 + \bar{H}_h - \bar{H}_{static} - \bar{C}_n \cdot \bar{q}_p^n). \quad (84)$$

Therefore, and assuming that $\bar{I}_H \neq 0$, the nominal pump head required to sustain the operating point is

$$\bar{H}_0 = \bar{C}_n \cdot \bar{q}_p^n - \bar{H}_h + \bar{H}_{static}. \quad (85)$$

Equation 85 provides a physically meaningful value for the controller bias. Accordingly, the PI bias is selected as $H_{bias} = \bar{H}_0$. This choice prevents the integral action from having to generate the complete nominal hydraulic requirement after controller initialization.

Consider small perturbations around the nominal operating point [63],

$$\begin{cases} q_p(t) = \bar{q}_p + \delta q_p(t) \\ H_0(t) = \bar{H}_0 + \delta H_0(t) \\ C_n(t) = \bar{C}_n + \delta C_n(t) \\ H_h(t) = \bar{H}_h + \delta H_h(t) \\ H_{static}(t) = \bar{H}_{static} + \delta H_{static}(t). \end{cases} \quad (86)$$

Starting from Eq. 81, the nonlinear resistive term is expanded to first order around $(\bar{q}_p \ \bar{C}_n)$:

$$C_n(t) \cdot q_p^n(t) \approx \bar{C}_n \cdot \bar{q}_p^n + n \cdot \bar{C}_n \cdot \bar{q}_p^{n-1} \cdot \delta q_p(t) + \bar{q}_p^n \cdot \delta C_n(t). \quad (87)$$

Substitution into Eq. 81, followed by cancellation of the nominal equilibrium terms using Eq. 85, gives [63]

$$\frac{d\delta q_p(t)}{dt} = -\frac{n \cdot \bar{C}_n \cdot \bar{q}_p^{n-1}}{\bar{I}_H} \cdot \delta q_p(t) + \frac{1}{\bar{I}_H} \cdot \delta H_0(t) - \frac{\bar{q}_p^n}{\bar{I}_H} \cdot \delta C_n(t) + \frac{1}{\bar{I}_H} \cdot \delta H_h(t) - \frac{1}{\bar{I}_H} \cdot \delta H_{static}(t) - \frac{1}{\bar{I}_H} \cdot \delta d_H(t). \quad (88)$$

This equation is useful since it explicitly shows which nonlinear plant variations are excluded from the nominal PI design but remain present in the nonlinear simulation.

A subtle point is also worth stating. Although $I_H(t)$ is time varying, its first-order perturbation does not appear explicitly in Eq. 88 when linearization is performed exactly at hydraulic equilibrium. Indeed, for

$$f(t) = \frac{F(t)}{I_H(t)}, \quad (89)$$

where

$$F(t) = H_0(t) + H_h(t) - H_{\text{static}}(t) - C_n(t) \cdot q_p^n(t) - d_H(t), \quad (90)$$

the derivative with respect to $I_H(t)$ is

$$\frac{\partial f(t)}{\partial I_H(t)} = -\frac{F(t)}{I_H^2(t)}. \quad (91)$$

Since $F(t)|_{\bar{p}} = 0$ it follows that

$$\left. \frac{\partial f(t)}{\partial I_H(t)} \right|_{\bar{p}} = 0. \quad (92)$$

Thus, changes in hydraulic inertance affect the full nonlinear process but do not enter the first-order nominal linearization directly at the selected equilibrium.

For PI synthesis, the perturbations associated with consistency, static head, level head, and external disturbances are neglected in the nominal design model,

$$\delta C_n(t) = \delta H_h(t) = \delta H_{\text{static}}(t) = \delta d_H(t) = 0. \quad (93)$$

Equation 88 therefore reduces to

$$\frac{\delta d_q}{dt} = -a_0 \cdot \delta q_p(t) + b_0 \cdot \delta H_0(t) \quad (94)$$

with

$$\begin{cases} a_0 = \frac{n \bar{C}_n \bar{q}_p^{n-1}}{\bar{I}_H} \\ b_0 = \frac{1}{\bar{I}_H} \end{cases} \quad (95)$$

Since $n > 0$, $C_n > 0$, $\bar{q}_p > 0$, $\bar{I}_H > 0$, it follows that $a_0 > 0$, $b_0 > 0$, and the locally linearized hydraulic subsystem is asymptotically stable [26].

Taking the Laplace transform (s -domain) of Eq. 94, assuming zero initial perturbation,

$$s \delta Q_p(s) = -a_0 \delta Q_p(s) + b_0 \delta H_0(s), \quad (96)$$

and therefore

$$G_q(s) = \frac{\delta Q_p(s)}{\delta H_0(s)} = \frac{b_0}{s + a_0}. \quad (97)$$

Equivalently,

$$G_q(s) = \frac{K_0}{\tau_0 s + 1}, \quad (98)$$

where the definition for K_0 and τ_0 is developed in

$$\begin{cases} \tau_0 = a_0 \Rightarrow \tau_0 = \frac{n \bar{C}_n \bar{q}_p^{n-1}}{\bar{I}_H} \\ K_0 = \frac{b_0}{a_0} \Rightarrow K_0 = \frac{1}{n \bar{C}_n \bar{q}_p^{n-1}} \end{cases} \quad (99)$$

The nominal nonlinear hydraulic plant is therefore locally represented by a stable first-order model. Reduced first-order models are widely used as the basis for analytically-derived PI tuning rules in industrial process control [62].

An important limitation follows directly from Eq. 99. For a shear-thinning exponent $0 < n < 1$,

$$\bar{q}_p^{n-1} = \frac{1}{\bar{q}_p^{1-n}}, \quad (100)$$

and the local model becomes increasingly sensitive as $\bar{q}_p \rightarrow 0$. Consequently, the PI controller is designed at a nonzero representative blowdown operating point, rather than near the terminal empty-digester condition.

Using the common tracking error described in Eq. 83, the unconstrained PI command is defined as

$$H_{0,PI}(t) = H_{\text{bias}} + K_p \cdot e_q(t) + x_1(t) \quad (101)$$

with

$$\frac{dx_1(t)}{dt} = K_i \cdot e_q(t), \quad (102)$$

where K_p and K_i are the proportional and integral gains, respectively, and $x_1(t)$ [m] represents the integral contribution to the commanded hydraulic head.

In the perturbation domain, assuming to have a unitary-feedback closed-control law, and according to Eq. 98, it follows

$$\delta H_0(s) = \left(K_p + \frac{K_i}{s} \right) \cdot \delta E_q(s), \quad (103)$$

therefore, that the PI transfer function is [63,64]:

$$C_{PI}(s) = K_p + \frac{K_i}{s} \Rightarrow C_{PI}(s) = \frac{K_p \cdot s + K_i}{s}. \quad (104)$$

Combining Eq. 97 and Eq. 104, the nominal closed-loop characteristic equation follows from

$$1 + C_{PI}(s) \cdot G_q(s) = 0. \quad (105)$$

Therefore,

$$1 + \left(\frac{K_p \cdot s + K_i}{s} \right) \cdot \left(\frac{b_0}{s + a_0} \right) = 0 \Rightarrow s^2 + (a_0 + b_0 \cdot K_p) \cdot s + b_0 \cdot (K_p \cdot s + K_i) = 0. \quad (106)$$

Hence [63,64],

$$s^2 + (a_0 + b_0 \cdot K_p) \cdot s + b_0 \cdot K_i = 0. \quad (107)$$

The addition of integral action therefore increases the nominal first-order process to a second-order closed-loop system.

PROCESS CONTROL

Pole-placement design is a well-established approach for PI/PID tuning when specified closed-loop transient characteristics are desired [65]. A desired second-order polynomial is selected as [63,64]:

$$s^2 + 2\cdot\zeta\cdot\omega_{cl}\cdot s + \omega_{cl}^2 = 0, \quad (108)$$

where $\zeta > 0$ [-] is the desired damping ratio and $\omega_{cl} > 0$ [rad/s] is the selected nominal closed-loop natural frequency. Also, the coefficient matching between Eq. 107 and Eq. 108 yields

$$\begin{cases} a_0 + b_0\cdot K_p = 2\cdot\zeta\cdot\omega_{cl} \\ b_0\cdot K_i = \omega_{cl}^2 \end{cases} \Rightarrow \begin{cases} K_p = \frac{2\cdot\zeta\cdot\omega_{cl} - a_0}{b_0} \\ K_i = \frac{\omega_{cl}^2}{b_0} \end{cases} \quad (109)$$

Using Eq. 95,

$$\begin{cases} K_p = 2\cdot\zeta\cdot\omega_{cl}\cdot\bar{I}_H - n\cdot\bar{C}_n\cdot\bar{q}_p^{n-1} \\ K_i = \omega_{cl}^2\cdot\bar{I}_H \end{cases} \quad (110)$$

For a conventional positive PI gain, $K_p > 0$ requires $2\cdot\zeta\cdot\omega_{cl} > a_0$. The selected ω_{cl} should nevertheless remain compatible with the available pump head range and with any neglected pump/drive dynamics.

Analytical PI tuning methods likewise emphasize the trade-off between desired response speed, robustness, and manipulated input activity [62].

The resulting nominal closed-loop transfer function is

$$\frac{\delta Q_p(s)}{\delta Q_p^*(s)} = \frac{b_0\cdot(K_p\cdot s + K_i)}{s^2 + (a_0 + b_0\cdot K_p)\cdot s + b_0\cdot K_i} \quad (111)$$

The integral term introduces infinite loop gain at zero frequency, and therefore provides zero nominal steady-state tracking error to a constant reference, provided that the closed loop is stable and actuator saturation is inactive:

Once Eq. 110 is evaluated at $(\bar{q}_p, \bar{C}_n, \bar{H}_h)$ the values of K_p and K_i remain fixed, i.e., $K_p(t) = K_p$ and $K_i(t) = K_i$. The actual nonlinear simulation, however, continues to evolve according to $(C_n(t), I_H(t), H_h(t), H_{static}(t))$. Thus, the discrepancy between the single nominal model used for PI synthesis and the continuously changing nonlinear plant constitutes part of the challenge imposed on the benchmark. This distinction is essential for a fair comparison with the SMC, which explicitly uses the nonlinear hydraulic structure.

The pump cannot generate an arbitrarily large hydraulic head. The unconstrained command in Eq. 101 is therefore limited according to

$$H_{0,PI}(t) = \text{sat}_{[H_{0,min}; H_{0,max}]}[H_{0,PI}^u(t)]. \quad (112)$$

Without additional compensation, continued integration while $H_{0,PI}(t) \neq H_{0,PI}^u(t)$ can produce integrator windup, re-

sulting in slower recovery and excessive transient response after the actuator leaves saturation. Integrator windup and anti-windup compensation for controllers subject to actuator constraints are well established in the control literature [63,66].

A back-calculation anti-windup term is therefore included into Eq. 102, giving

$$\frac{dx_1(t)}{dt} = K_i\cdot e_q(t) + K_{aw}\cdot (H_{0,PI}(t) - H_{0,PI}^u(t)), \quad (113)$$

where $K_{aw} > 0$ [s⁻¹] is the anti-windup gain.

When the actuator is unsaturated, $H_{0,PI}(t) = H_{0,PI}^u(t)$, Eq. 113 reduces to conventional integral law, i.e.,

$$\frac{dx_1(t)}{dt} = K_i\cdot e_q(t). \quad (114)$$

During saturation, however, the correction term drives the controller integral state toward a value consistent with the realizable pump head command. Back-calculation/conditioning strategies of this general type are part of the established anti-windup framework for linear controllers with actuator nonlinearities [63].

Thus, the final fixed-gain is

$$\begin{cases} e_q(t) = q_p^*(t) - q_p(t) \\ H_{0,PI}^u(t) = \bar{H}_0 + K_p\cdot e_q(t) + x_1(t) \\ H_{0,PI}(t) = \text{sat}_{[H_{0,min}; H_{0,max}]}[H_{0,PI}^u(t)] \\ \frac{dx_1(t)}{dt} = K_i\cdot e_q(t) + K_{aw}\cdot (H_{0,PI}(t) - H_{0,PI}^u(t)) \end{cases} \quad (115)$$

Nonlinear sliding-mode flow controller

The SMC is synthesized directly from the nonlinear hydraulic dynamics developed in Eq. 81, thereby retaining the time-varying hydraulic inertance, consistency-dependent resistance, digester-level contribution, and density-dependent static head in the control law. Sliding mode control is well suited to nonlinear time-varying systems subject to bounded disturbances and parameter variations, and its robustness has been demonstrated experimentally in nonlinear hydraulic systems [67].

For compactness, define the nonlinear hydraulic head requirement as

$$\Phi(t) = C_n(t)\cdot q_p^n(t) + H_{static}(t) - H_h(t), \quad (116)$$

and therefore, the modified version of Eq. 81 may be Eq. 117. Using the common tracking-error definition in Eq. 83, an integral-augmented sliding variable is selected as Eq. 118, where $\xi_q(t)$ [m³] is the integral tracking-error state and $\lambda_q > 0$ [s⁻¹] determines the error-convergence rate once sliding motion is established:

$$I_H(t) \cdot \frac{dq_p(t)}{dt} = H_0(t) - \Phi(t) - d_H(t). \quad (117)$$

$$\begin{cases} s_q(t) = e_q(t) + \lambda_q \cdot \xi_q(t) \\ \frac{d\xi_q(t)}{dt} = e_q(t). \end{cases} \quad (118)$$

Integral sliding structures have been applied successfully to hydraulic systems subject to unknown disturbances and parameter uncertainty.

Differentiating Eq. 118 gives

$$\frac{ds_q(t)}{dt} = \frac{dq_p^*(t)}{dt} - \frac{dq_p(t)}{dt} + \lambda_q \cdot e_q(t). \quad (119)$$

Substituting Eq. 117,

$$\frac{ds_q(t)}{dt} = \frac{dq_p^*(t)}{dt} - \frac{1}{I_H(t)} \cdot (H_0(t) - \Phi(t) - d_H(t)) + \lambda_q \cdot e_q(t). \quad (120)$$

For the nominal model, the unknown disturbance is neglected only in the derivation of the equivalent control, i.e., $d_H(t) = 0$. Imposing the nominal sliding condition valid for $I_H(t) \neq 0$,

$$\frac{ds_q(t)}{dt} = \frac{dq_p^*(t)}{dt} - \frac{1}{I_H(t)} \cdot (H_0(t) - \Phi(t) - d_H(t)) + \lambda_q \cdot e_q(t). \quad (121)$$

Using Eq. 116,

$$H_{0,eq}(t) = C_n(t) \cdot q_p^n(t) + H_{static}(t) - H_h(t) + I_H(t) \cdot \left(\frac{dq_p^*(t)}{dt} + \lambda_q \cdot e_q(t) \right). \quad (122)$$

This equation is important since it separates the controller into physically interpretable contributions. The term $C_n(t) \cdot q_p^n(t)$ compensates the instantaneous nonlinear slurry transport resistance and $H_{static}(t) - H_h(t)$ provides the hydraulic acceleration required to impose the desired tracking dynamics.

For a constant or piecewise-constant discharge flow reference,

$$\frac{dq_p^*(t)}{dt} = 0 \quad (123)$$

during the constant-reference portions of the blowdown cycle, and therefore

$$H_{0,eq}(t) = C_n(t) \cdot q_p^n(t) + H_{static}(t) - H_h(t) + I_H(t) \cdot \lambda_q \cdot e_q(t). \quad (124)$$

For discontinuous changes in $q_p^*(t)$, a filtered or rate-limited command should be employed so that $\frac{dq_p^*(t)}{dt}$ remains finite.

When the trajectory lies on the sliding manifold,

$$\frac{ds_q(t)}{dt} = 0, \quad (125)$$

and from Eq. 119,

$$\frac{de_q(t)}{dt} + \lambda_q \cdot e_q(t) = 0 \Rightarrow e_q(t) = e_q(t_s) \cdot e^{-\lambda_q \cdot (t-t_s)}, \quad t \geq t_s, \quad (126)$$

where t_s is the reaching time. Thus, λ_q has a direct control interpretation: increasing λ_q increases the nominal convergence rate of the flow tracking error after the sliding manifold has been reached.

A robust switching contribution is added to compensate for model uncertainty and external hydraulic disturbances:

$$H_{0,SMC}^u(t) = H_{0,eq}(t) + k_H \cdot \text{sat}\left(\frac{s_q(t)}{\phi_q}\right), \quad (127)$$

where $k_H > 0$ [m] is the robust switching head gain and $\phi_q > 0$ [m³/s] defines the boundary layer thickness.

The positive sign in front of the switching term is consistent with the error convention in Eq. 83. For instance, if $q_p(t) < q_p^*(t)$, then $e_q(t) > 0$, normally $s_q(t) > 0$, and the switching contribution increases $H_0(t)$, thereby increasing the hydraulic driving force and pushing $q_p(t)$ toward its reference.

Replacing the ideal sign function by a continuous boundary layer is a classical SMC technique for reducing chattering while retaining practical robustness [31].

For numerical implementation, the switching action may alternatively be smoothed using

$$H_{0,SMC}^u(t) = H_{0,eq}(t) + k_H \cdot \tanh\left(\frac{s_q(t)}{\phi_q}\right). \quad (128)$$

The physical pump head command is then defined and constrained by

$$H_{0,SMC}(t) = \text{sat}_{[H_0, \min; H_0, \max]}[H_{0,SMC}^u(t)]. \quad (129)$$

Let $\hat{I}_H(t)$ and $\hat{\Phi}_H(t)$ denote the model quantities available to the controller. Taking into account Eq. 116 and Eq. 122, the implemented equivalent control is

$$H_{0,eq}(t) = \hat{\Phi}(t) + \hat{I}_H(t) \cdot \left(\frac{dq_p^*(t)}{dt} + \lambda_q \cdot e_q(t) \right). \quad (130)$$

PROCESS CONTROL

Define

$$v_q(t) = \frac{dq_p^*(t)}{dt} + \lambda_q \cdot e_q(t). \quad (131)$$

Substituting Eq. 128 and Eq. 130 into Eq. 117, and after some mathematical operations, yields:

$$I_H(t) \cdot \frac{ds_q(t)}{dt} = \Delta_H(t) - k_H \cdot \text{sat}\left(\frac{s_q(t)}{\phi_q}\right), \quad (132)$$

where the equivalent matched hydraulic-head uncertainty is:

$$\Delta_H(t) = (I_H(t) - \hat{I}_H(t)) \cdot v_q(t) + (\Phi(t) - \hat{\Phi}(t)) + d_H(t). \quad (133)$$

Thus, $\Delta_H(t)$ combines uncertainties arising from the slurry-dependent hydraulic inertance, nonlinear resistance, static and level head reconstruction, and unresolved external hydraulic disturbances.

This treatment is aligned with hydraulic sliding mode literature, where nonlinear parameter uncertainty, changing hydraulic properties, and external disturbances are explicitly addressed through robust or adaptive SMC terms [68].

Assume that $0 \leq I_{H,\min} \leq I_{H,\max}$ and that $|\Delta_H(t)| \leq \bar{\Delta}_H(t)$. Consider the Lyapunov candidate

$$V_s(t) = 0.5 \cdot s_q^2(t). \quad (134)$$

Its time derivative is $\frac{dV_s(t)}{dt} = s_q(t) \cdot \frac{ds_q(t)}{dt}$. Outside the boundary layer, $|s_q(t)| > \phi_q$, therefore $\text{sat}\left(\frac{s_q(t)}{\phi_q}\right) = \text{sgn}(s_q(t))$. Using Eq. 132,

$$\frac{dV_s(t)}{dt} = \frac{s_q(t)}{I_H(t)} \cdot \left(\Delta_H(t) - k_H \cdot \text{sgn}\left(\frac{s_q(t)}{\phi_q}\right) \right). \quad (135)$$

Since

$$s_q(t) \cdot \Delta_H(t) \leq |s_q(t)| \cdot |\Delta_H(t)| \leq \bar{\Delta}_H(t) \cdot |s_q(t)|, \quad (136)$$

it follows that:

$$\frac{dV_s(t)}{dt} \leq -\frac{k_H - \bar{\Delta}_H(t)}{I_H(t)} \cdot |s_q(t)|. \quad (137)$$

Therefore:

$$\frac{dV_s(t)}{dt} \leq -\frac{k_H - \bar{\Delta}_H(t)}{I_{H,\max}} \cdot |s_q(t)|. \quad (138)$$

A sufficient reaching condition is consequently:

$$k_H(t) > \bar{\Delta}_H(t) \quad (139)$$

Under this condition, $\frac{dV_s(t)}{dt} \leq 0$, and the trajectory

is driven toward the sliding manifold. This type of Lyapunov-based robustness argument is standard in sliding mode control and is also used in nonlinear hydraulic SMC formulations with uncertain parameters [68,69].

Inside the boundary layer, $|s_q(t)| \leq \phi_q$, and $\left(\frac{s_q(t)}{\phi_q}\right) = \frac{s_q(t)}{\phi_q}$. Equation 132 becomes:

$$I_H(t) \cdot \frac{ds_q(t)}{dt} = \Delta_H(t) - \left(\frac{k_H}{\phi_q}\right) \cdot s_q(t) \quad (140)$$

At the limiting quasi-steady condition, $\frac{ds_q(t)}{dt} = 0$, the residual sliding variable satisfies

$$|s_q(t)| \leq \frac{\phi_q \cdot \bar{\Delta}_H(t)}{k_H}. \quad (141)$$

This relation explicitly shows the boundary layer trade-off: a smaller ϕ_q improves tracking precision but increases switching activity, whereas a larger ϕ_q produces smoother control action at the expense of a larger quasi-sliding neighborhood. Classical sliding mode work identifies precisely this compromise between robustness, tracking accuracy, and chattering mitigation [67,69].

Pump actuation and terminal interlock

The controller design treats $H_0(t)$ as the effective manipulated variable at the hydraulic model interface. In an actual variable-speed pumping installation, the physical actuator would normally be the motor drive speed or frequency, which modifies the pump head and consequently the delivered flow. Gibson [70] explicitly discusses variable-speed centrifugal pumps as flow control elements and emphasizes the interaction between the pump H-Q characteristic and the process characteristic.

Finite pump/drive response is represented by the reduced first-order actuator model

$$\tau_H \cdot \frac{dH_0(t)}{dt} + H_0(t) = H_{0,c}(t), \quad (142)$$

where $\tau_H > 0$ [s] is the effective pump head response time constant, and $H_{0,c}(t)$ is generated by either the PI or SMC controller.

For the SMC,

$$H_{0,c}(t) = \text{sat}_{[H_{0,\min}; H_{0,\max}]}[H_{0,SMC}^u(t)]. \quad (143)$$

The Lyapunov reaching condition derived above assumes that sufficient control authority is available. A useful sufficient condition for avoiding saturation of the complete switching contribution is therefore:

$$H_{0,\min} + k_H \leq H_{0,eq}(t) \leq H_{0,\max} - k_H. \quad (144)$$

If this condition is violated, the physical actuator cannot

supply the full switching action, and the ideal reaching condition cannot be guaranteed.

The theoretical derivation also treats the generated pump head $H_0(t)$ as the direct hydraulic input. When the actuator dynamics in Eq. 142 are non-negligible, the mismatch $H_{c,0}(t) - H_0(t)$ acts as an additional matched hydraulic uncertainty. Consequently, the outer-loop SMC design assumes that the pump/drive dynamics are sufficiently faster than the dominant blow line dynamics, or that the resulting actuator error remains bounded within the uncertainty margin used for selecting k_H .

Finally, when the remaining dry fiber mass, total slurry inventory, or equivalent slurry level falls below its prescribed minimum admissible threshold, the effective discharge reference is set to zero, the pump command is removed, and the controller integral states are frozen. This terminal interlock prevents either controller from attempting to maintain a nonzero blowdown flow after the available slurry inventory has effectively been exhausted.

SIMULATION RESULTS

The complete nonlinear closed-loop model was simulated for both the fixed-gain PI and SMC strategies under identical initial conditions, actuator constraints, reference trajec-

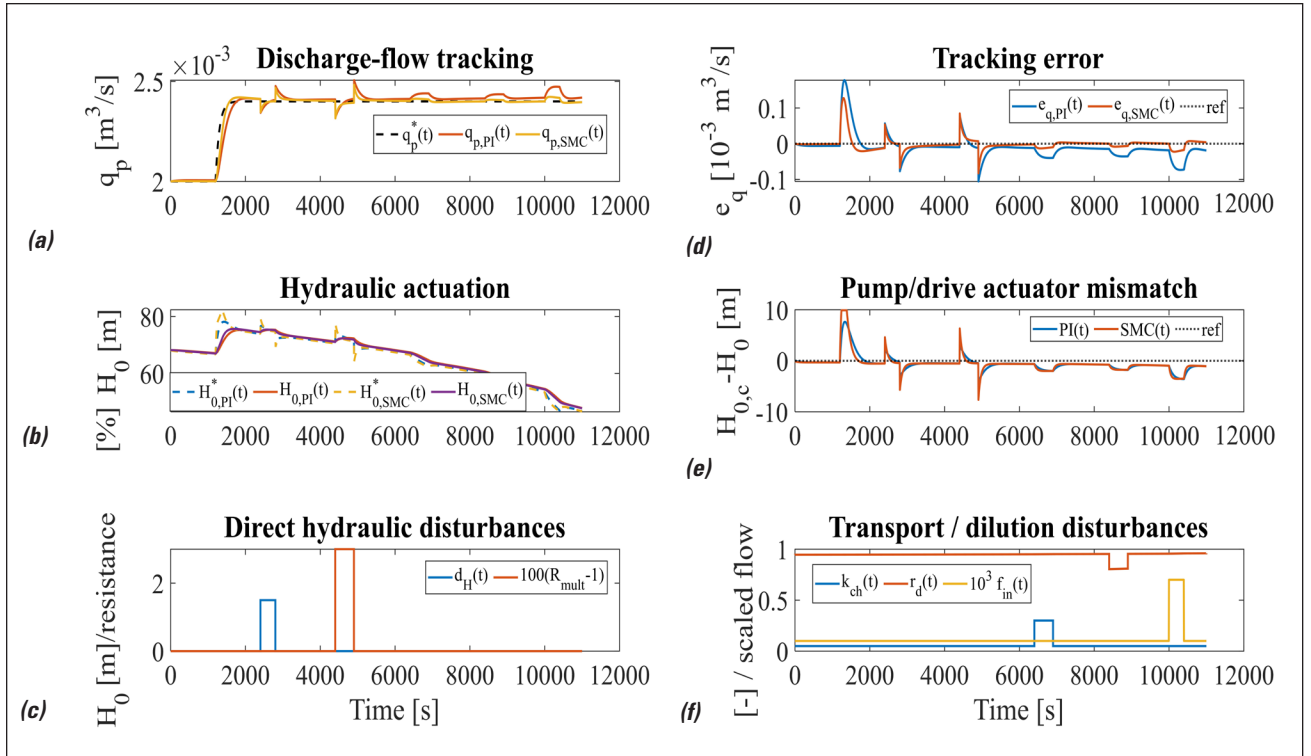
tories, and disturbance sequences. The equations were integrated using MATLAB ode15s (MathWorks; Natick, MA, USA) with a relative tolerance of 10^{-6} , individual absolute tolerances for the state variables, and a maximum integration step of 20 s. **Table I** summarizes the simulation parameters.

Figure 3 presents the direct PI–SMC comparison. Figure 3a shows the discharge-flow tracking responses, where both controllers follow the prescribed reference. Figure 3b presents the commanded and realized pump-head responses, illustrating the hydraulic actuation required by each controller. Figure 3c shows the imposed direct hydraulic disturbances. Figure 3d presents the corresponding tracking error, where the SMC generally exhibits smaller excursions following the imposed perturbations. Figure 3e shows the pump/drive actuator mismatch, highlighting the effect of the finite actuator dynamics. Finally, Fig. 3f presents the transport and dilution disturbances, which are temporally separated to allow their individual closed-loop effects to be distinguished.

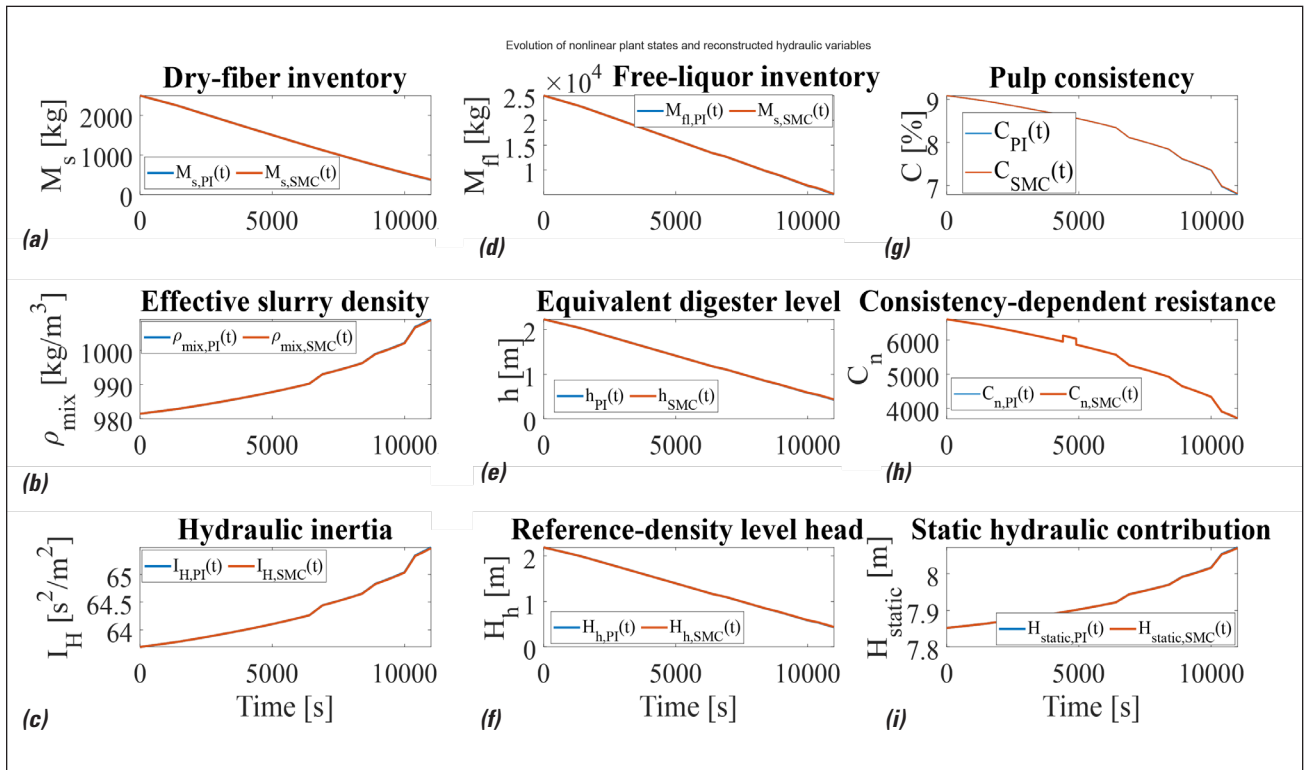
The evolution of the underlying nonlinear plant is shown in **Fig. 4**. Dry fiber inventory decreases monotonically in Fig. 4a, while the effective slurry density progressively changes in Fig. 4b, producing the corresponding variation

Parameter	Description	Value	Unit
ρ_s	Effective dry fiber density	1050	kg/m ³
ρ_{fl}	Free liquor density	1100	kg/m ³
D_d	Equivalent digester diameter	4.0	m
D_p	Blow line diameter	0.20	m
L_{eff}	Effective hydraulic length	20	m
C^*	Reference consistency	0.10	–
K^*	Reference hydraulic resistance	8.0×10^3	(m ³ /s) ^{n}
α_c	Consistency resistance exponent	2.0	–
n	Nominal nonlinear flow exponent	0.75	–
$M_s(0)$	Initial dry fiber mass	2500	kg
$M_{fl}(0)$	Initial free liquor mass	25000	kg
$H_{0,max}$	Maximum pump head	120	m
τ_H	Pump/drive time constant	300	s
q_p^*	Principal discharge flow reference	2.4×10^{-3}	m ³ /s
λ_q	Sliding-manifold gain	1.0×10^{-3}	s ⁻¹
k_H	Robust SMC head gain	10	m
ϕ_q	Boundary-layer thickness	1.0×10^{-4}	m ³ /s

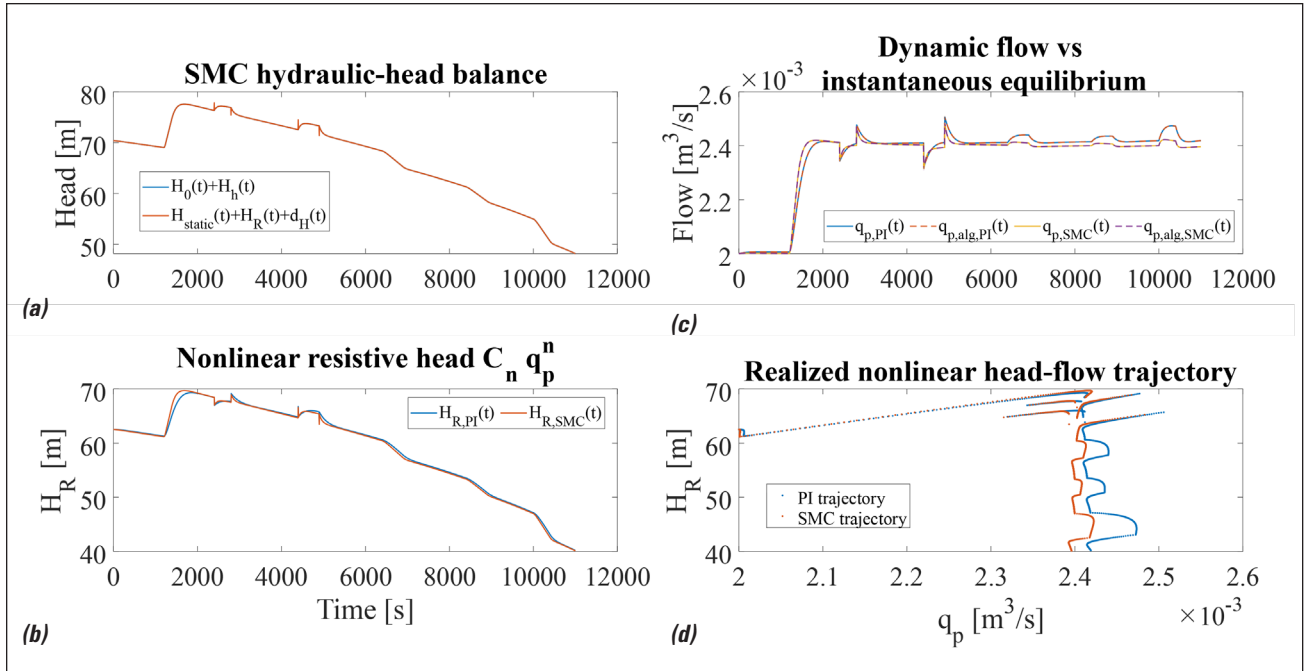
I. Simulation process variable.



3. The proportional-integral (PI) controller and SMC comparison for nonlinear batch digester discharge: (a) discharge-flow tracking, (b) hydraulic actuation, (c) direct hydraulic disturbances, (d) tracking error, (e) pump/drive actuator mismatch, and (f) transport/dilution disturbances.



4. Evolution of the main nonlinear batch digester states and reconstructed hydraulic variables under PI and SMC control: (a) dry fiber inventory, (b) slurry density, (c) hydraulic inertia, (d) free liquor inventory, (e) digester level, (f) level head, (g) pulp consistency, (h) consistency-dependent resistance, and (i) static hydraulic head.



5. Nonlinear hydraulic behavior under PI and SMC control: (a) SMC hydraulic head balance, (b) nonlinear resistive head, (c) dynamic flow vs. instantaneous equilibrium flow, and (d) realized nonlinear head flow trajectories.

in hydraulic inertance shown in Fig. 4c. Free liquor depletion is shown in Fig. 4d, together with the associated reduction in equivalent digester level in Fig. 4e and reference-density level head in Fig. 4f. The resulting consistency evolution in Fig. 4g modifies the nonlinear resistance $C_n(t)$, as observed in Fig. 4h, while the density dependence of the static contribution appears in Fig. 4i. The close PI–SMC overlap of these slowly varying states confirms that both controllers are compared over essentially the same evolving batch condition.

Figure 5 verifies the principal hydraulic relationships developed in the “System Modeling” section. The SMC head decomposition in Fig. 5a shows the instantaneous balance between driving and opposing hydraulic contributions. Figure 5b illustrates the strongly time-varying nonlinear resistive head $H_R(t) = C_n(t) \cdot q_p^n(t)$. The comparison between actual and instantaneous equilibrium flow in Fig. 5c demonstrates the finite inertance-induced hydraulic response predicted by Eq. 65, while the trajectories in Fig. 5d confirm that the process continuously migrates through a nonlinear $(q_p(t), H_R(t))$ operating domain rather than remaining near a single equilibrium point, even using PI and SMC.

The energetic response is summarized in **Fig. 6**. Electrical input power is presented in Fig. 6a, and its integration yields the cumulative electrical energy in Fig. 6b. The resistive head fraction $\chi_R(t)$ in Fig. 6c quantifies the partition of the hydraulic requirement associated with nonlinear slurry resistance, while Fig. 6d shows the corresponding transport resistance power $P_R(t)$. The batch-specific dry fiber SEC is reported in Fig. 6e. The model-based transport indicator in Fig. 6f is used only as a reduced-order hydraulic perfor-

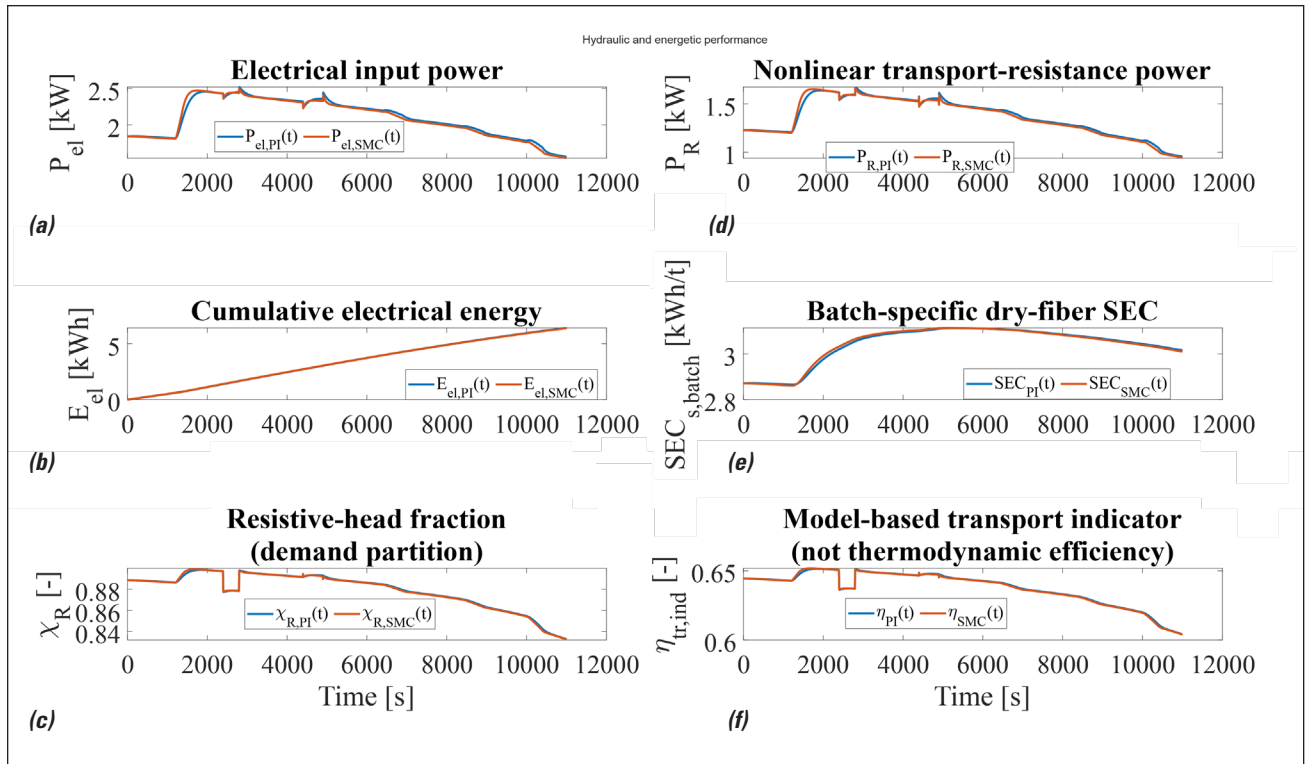
mance metric and is not interpreted as a thermodynamic efficiency.

The disturbance sequence used for the robustness assessment is detailed separately in **Fig. 7**. The additive opposing-head pulse is shown in Fig. 7a, channeling variation in Fig. 7b, dilution-liquor pulse in Fig. 7c, unmodeled resistance multiplier in Fig. 7d, and drainability perturbation in Fig. 7e. Their corresponding channeling and drainability correction factors are shown in Fig. 7f. The temporal separation between disturbances provides sufficient recovery intervals to distinguish their individual closed-loop effects.

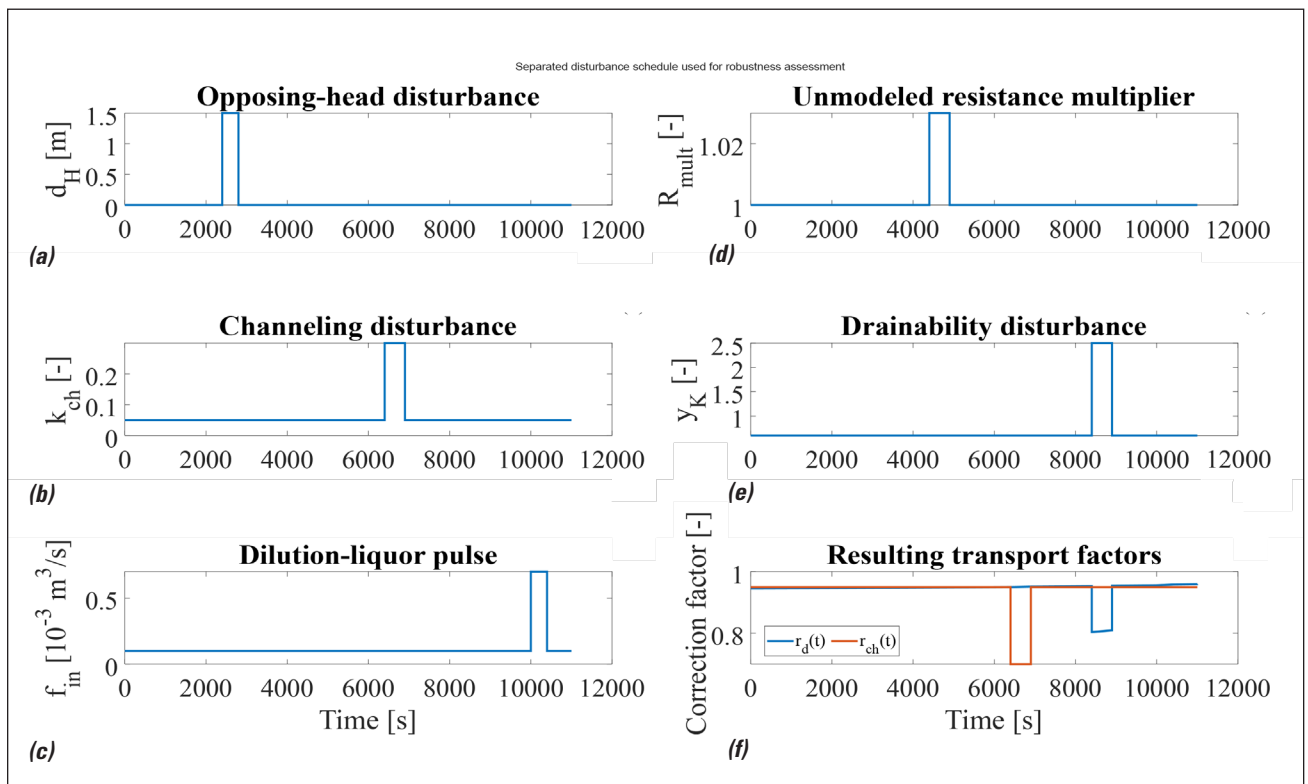
The nonlinear dependence of hydraulic performance on $q_p(t)$ and $C_n(t)$ is visualized in **Fig. 8**. The three-dimensional model-based transport indicator surface is shown in Fig. 8a, whereas Fig. 8b provides the corresponding two-dimensional contour representation and superimposes the realized PI and SMC trajectories.

The electrical energy requirement exhibits an even stronger nonlinear dependence on operating point. **Figure 9a** presents the model-based electrical power surface, while Fig. 9b shows its two-dimensional contour together with both closed-loop trajectories. These plots demonstrate that increasing flow and hydraulic resistance progressively increases the required electrical power, while the actual trajectories occupy only a restricted portion of the feasible operating domain.

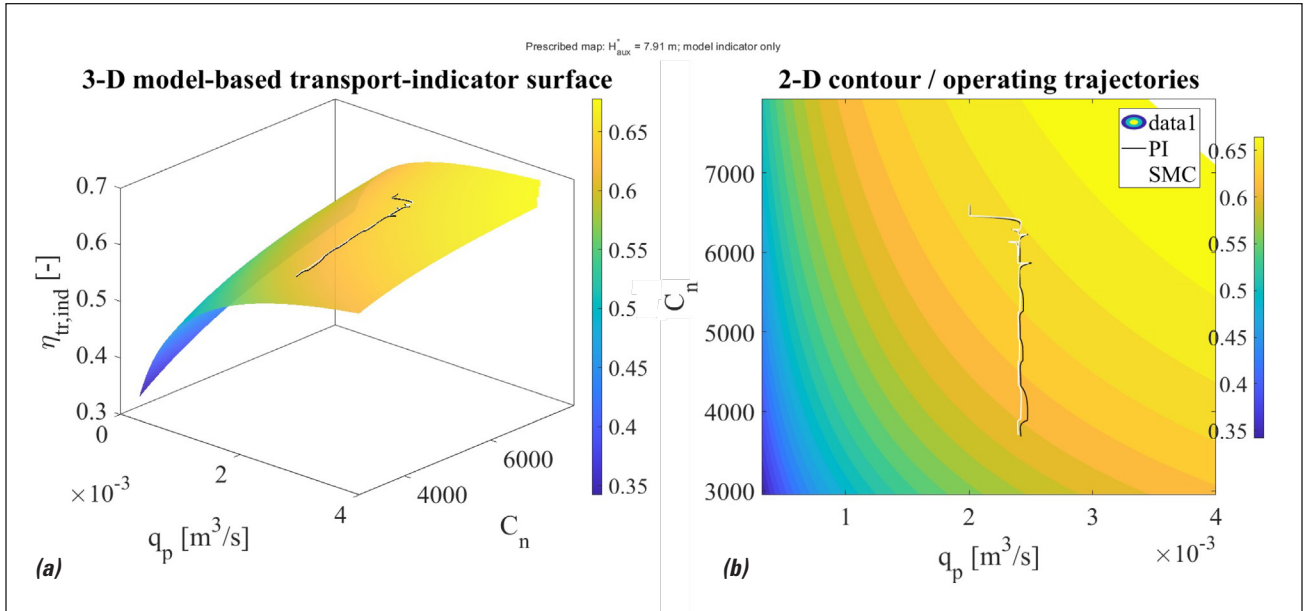
The internal behavior of the SMC is presented in **Fig. 10**. The sliding variable in Fig. 10a repeatedly returns toward the prescribed boundary layer after reference and disturbance events. The $(e_q(t), \xi_q(t))$ trajectory in Fig. 10b approaches the ideal manifold $s_q(t) = 0$, while the three-



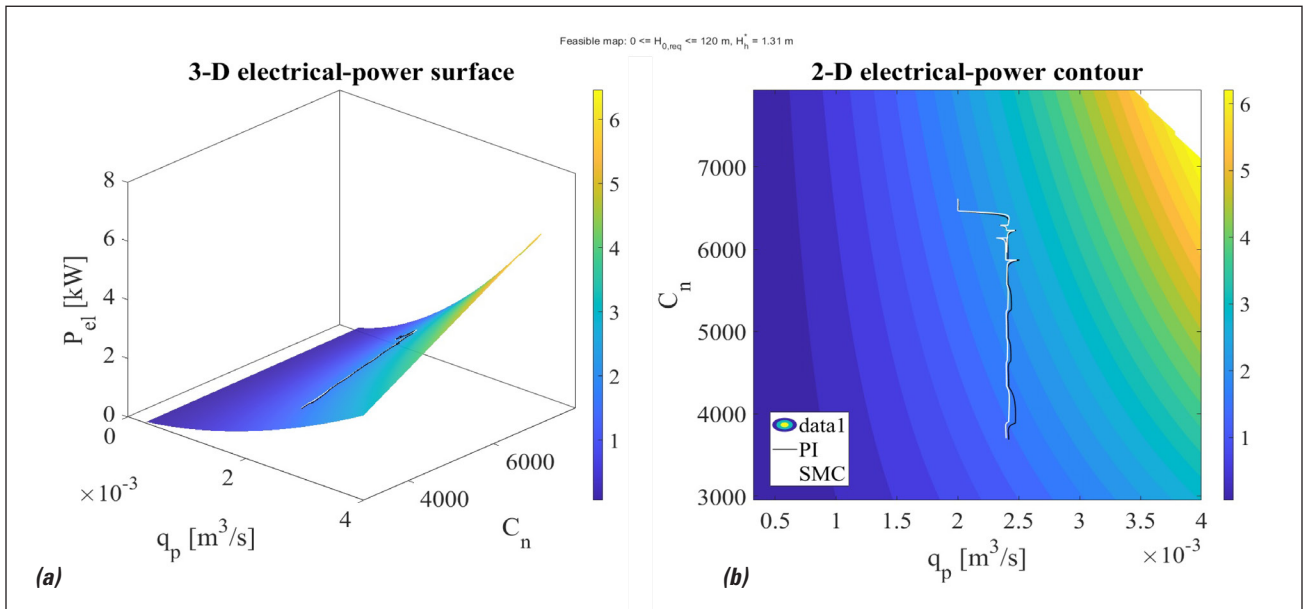
6. Energetic comparison of PI and SMC operation: (a) electrical input power, (b) cumulative electrical energy, (c) resistive head fraction, (d) nonlinear transport resistance power, (e) batch-specific dry fiber specific energy consumption (SEC), and (f) model-based transport indicator.



7. Applied disturbance profiles: (a) opposing head disturbance, (b) channeling disturbance, (c) dilution-liquor pulse, (d) unmodeled resistance variation, (e) drainability disturbance, and (f) resulting transport correction factors.



8. Model-based transport indicator maps: (a) three-dimensional surface as a function of discharge flow $q_p(t)$ and nonlinear resistance $C_n(t)$, and (b) corresponding 2-D contour map with the realized PI and SMC operating trajectories.



9. Model-based electrical power maps: (a) 3-D electrical power surface as a function of discharge flow $q_p(t)$ and nonlinear resistance $C_n(t)$, and (b) corresponding 2-D power contour with the realized PI and SMC operating trajectories.

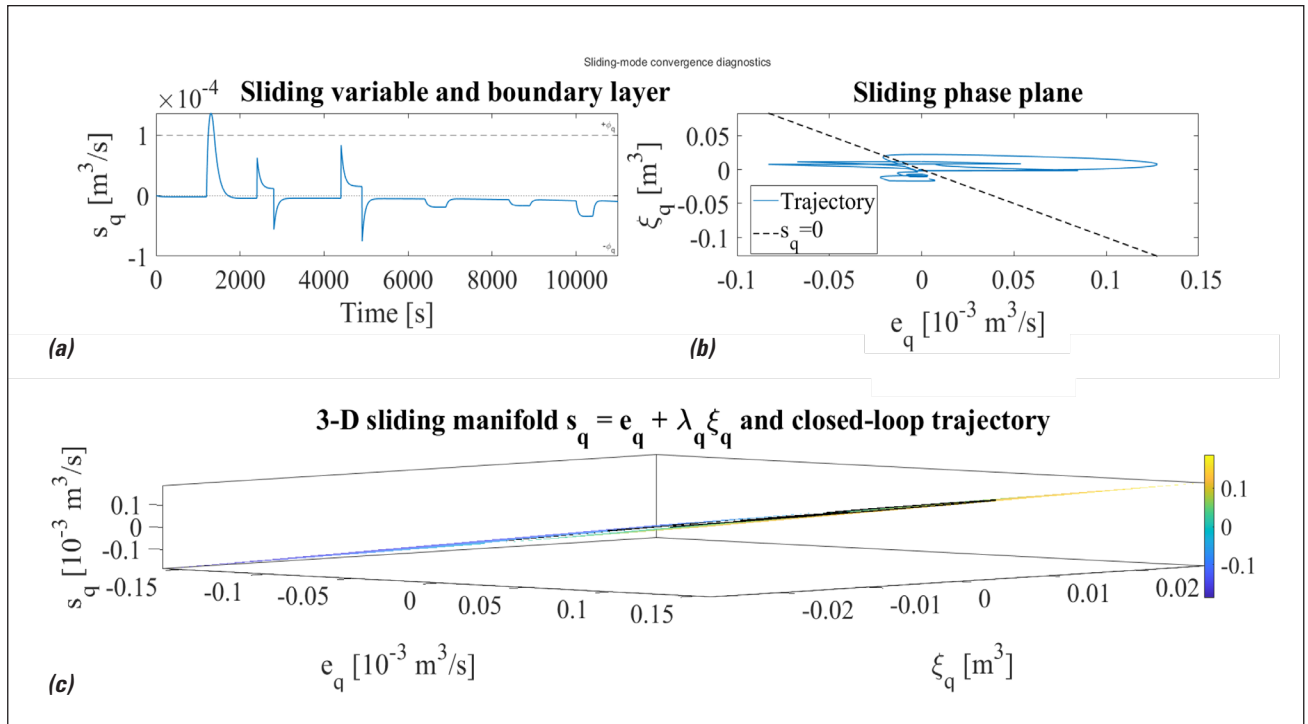
dimensional representation in Fig. 10c shows the realized closed-loop trajectory relative to Eq. 118.

Accordingly, the numerical response is consistent with practical boundary-layer sliding, rather than ideal discontinuous sliding.

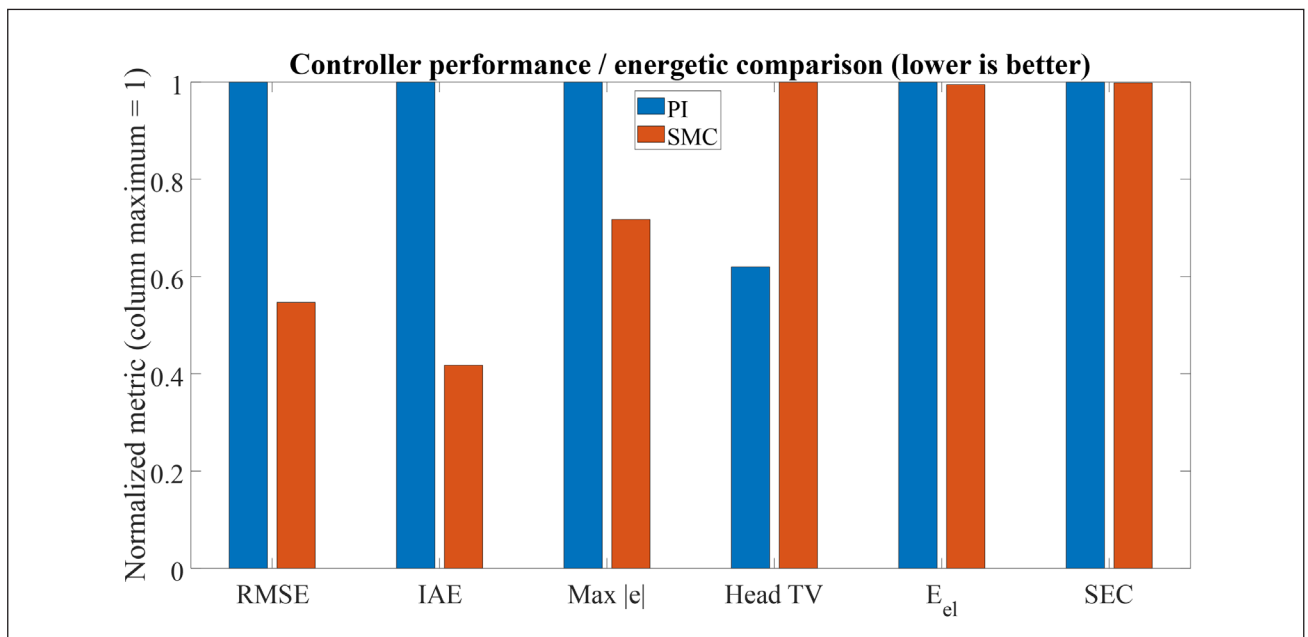
The normalized performance comparison in **Fig. 11** confirms that the SMC provides lower root-mean-square error (RMSE), integral absolute error (IAE), and maximum tracking error than the fixed-gain PI. This improvement is obtained at the expense of greater hydraulic head total variation, whereas cumulative electrical energy and

batch-specific SEC remain comparatively similar. Thus, the principal SMC advantage is improved tracking robustness rather than a substantial reduction in total batch energy.

Finally, robustness against rheological uncertainty is examined by varying the true nonlinear exponent n while retaining the controllers designed at the nominal value. **Figure 12** shows the progressive influence of n on the fixed-gain PI flow response. The quantitative comparison in **Fig. 13** shows the resulting RMSE in Fig. 13a, electrical energy in Fig. 13b, the IAE in Fig. 13c, and batch-specific



10. Sliding-mode behavior of the SMC: (a) sliding variable with boundary layer, (b) sliding phase-plane trajectory and ideal manifold $s_q(t) = 0$, and (c) three-dimensional sliding manifold with the realized closed-loop trajectory.

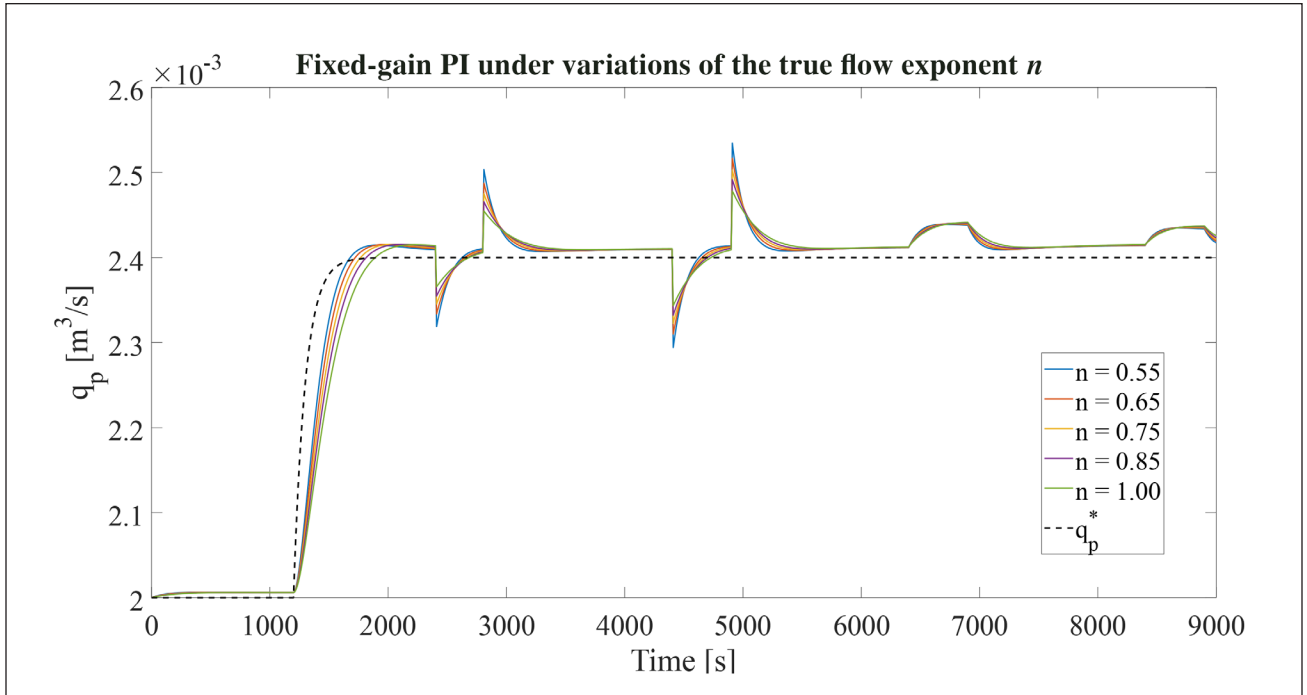


11. Normalized PI-SMC performance and energetic comparison using root-mean-square error (RMSE), integral absolute error (IAE), maximum tracking error, hydraulic head total variation, cumulative electrical energy, and batch-specific specific energy consumption (SEC); lower values indicate better performance.

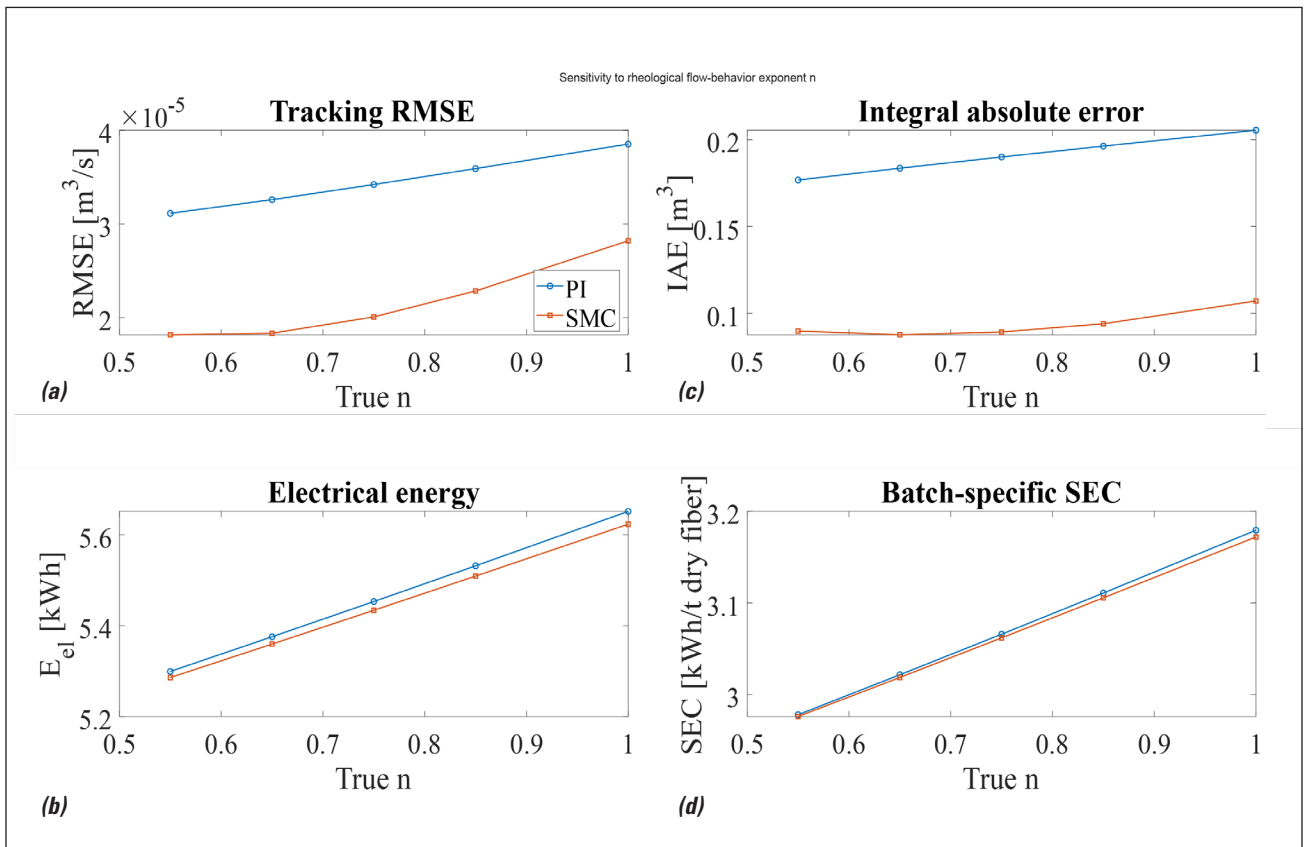
SEC in Fig. 13d. The SMC maintains lower tracking error indices throughout the investigated n -range, whereas the differences in electrical energy and SEC remain small, demonstrating that uncertainty in n primarily affects regulation quality rather than the overall energy requirement.

The 11000-s horizon was intentionally selected as an

extended numerical robustness test, rather than as a representation of the physical duration of a single industrial digester blow. The extended interval provides clear temporal separation between successive disturbances, allows recovery of the relatively slow pump/drive dynamics ($\tau_H = 300$ s), and makes cumulative inventory and energetic ef-



12. Sensitivity of the fixed-gain PI controller to variations in the nonlinear flow exponent n , showing discharge flow tracking for $n = 0.55$ – 1.00 relative to the prescribed reference $q_p^*(t)$.



13. Effect of the true nonlinear flow exponent n on PI and SMC performance: (a) tracking RMSE, (b) electrical energy consumption, (c) IAE, and (d) batch-specific SEC.

fects observable. Industrial batch digester blows themselves are typically much shorter; historical industrial guidance reports approximately 10–20 min, including an example of a 200-m³ digester discharged over approximately 20 min.

CONCLUSIONS

A nonlinear control-oriented framework was developed for batch digester blowdown by coupling dry fiber and free liquor inventories with evolving consistency, mixture density, digester level, hydraulic inertance, channeling, drainability, and consistency-dependent nonlinear resistance. The resulting model reproduces the strongly time-varying hydraulic operating conditions expected for concentrated pulp suspensions, whose rheology is known to depend strongly on fiber interactions and concentration.

Under identical nonlinear plant conditions and sequential disturbances, the SMC consistently achieved lower RMSE, IAE, and peak flow tracking error than the fixed-gain PI benchmark, with rapid recovery toward the prescribed sliding boundary layer. This improved regulation was obtained at the expense of higher hydraulic control activity, whereas cumulative electrical energy and dry fiber-specific SEC remained comparatively similar. The results therefore indicate that the primary advantage of the proposed SMC is robust hydraulic regulation rather than substantial energy reduction, consistent with the established robustness of sliding mode methods for uncertain nonlinear hydraulic systems.

Sensitivity analysis with respect to the nonlinear exponent n further showed that the SMC maintains superior tracking robustness over the investigated rheological range. The proposed hydraulic demand, power, SEC, and sliding manifold representations additionally provide physically interpretable tools for characterizing the evolving nonlinear operating domain; SEC itself is an established metric for assessing slurry transport energetic performance. Overall, the results support the proposed formulation as a proof-of-concept nonlinear modeling and robust control framework, while experimental parameter identification and plant-scale validation remain necessary before industrial implementation.

LITERATURE CITED

1. Bijok, N., Fiskari, J., Gustafson, R.R., et al., *Chem. Eng. Res. Des.* 193: 13(2023). <https://doi.org/10.1016/j.cherd.2023.03.010>.
2. Ek, M., Gellerstedt, G., and Henriksson, G., Eds., *Pulping Chemistry and Technology*, Pulping Chemistry and Technology Series, Vol. 2, De Gruyter, Berlin, 2009. <https://doi.org/10.1515/9783110213423>.
3. Empie, H.J., *Fundamentals of the Kraft Recovery Process*, TAPPI Press, Atlanta, 2009.
4. Vanchinathan, S. and Krishnagopalan, G.A., *TAPPI J.* 80(3): 123(1997).

5. Tikka, P., Ed., *Chemical Pulping, Part 2—Recovery of Chemicals and Energy*, Papermaking Science and Technology, Book 6, 2nd edn., Finnish Paper Engineers' Association/Paperi ja Puu, Helsinki, 2008.
6. Castro, P., Matos, H., and Barbosa-Póvoa, A.P.F.D., *Comput. Chem. Eng.* 26(4): 671(2002). [https://doi.org/10.1016/S0098-1354\(01\)00792-X](https://doi.org/10.1016/S0098-1354(01)00792-X).
7. Derakhshandeh, B., Kerekes, R.J., Hatzikiriakos, S.G., et al., *Chem. Eng. Sci.* 66(15): 3460(2011). <https://doi.org/10.1016/j.ces.2011.04.017>.
8. Campos-Salazar, J.M., Santander, F., and Larraín, S., "Dynamic modeling and control system analysis for continuous-disc filters in pulp mill operations," *Int. Chem. Recovery Conf.*, TAPPI/PAPTAC, Peachtree Corners, GA, USA/Brossard, QC, Canada, 2025. <https://doi.org/10.48550/arXiv.2510.02385>
9. Bennington, C.P.J., Kerekes, R.J., and Grace, J.R., *Can. J. Chem. Eng.* 68(5): 748(1990). <https://doi.org/10.1002/cjce.5450680503>.
10. Pike, R.W., *Optimization for Engineering Systems*, Van Nostrand Reinhold, New York, 1986.
11. Kerekes, R., *Nord. Pulp Pap. Res. J.* 21(5): 598(2006). <https://doi.org/10.3183/NPPRJ-2006-21-05-p598-612>.
12. Singh, R., Lavrykov, S., and Ramarao, B.V., *Colloids Surf., A* 333(1-3): 96(2009). <https://doi.org/10.1016/j.colsurfa.2008.09.035>.
13. Pal, R., *Adv. Colloid Interface Sci.* 333: 103299(2024). <https://doi.org/10.1016/j.cis.2024.103299>.
14. Petrie, C.J.S., *J. Non-Newtonian Fluid Mech.* 87(2-3): 369(1999). [https://doi.org/10.1016/S0377-0257\(99\)00069-5](https://doi.org/10.1016/S0377-0257(99)00069-5).
15. Ventura, C., Garcia, F., Ferreira, P., et al., *TAPPI J.* 7(8): 20(2008). <https://doi.org/10.32964/TJ7.8.20>.
16. Chhabra, R.P. and Richardson, J.F., *Non-Newtonian Flow and Applied Rheology: Engineering Applications*, 2nd edn., Butterworth-Heinemann, Oxford, UK, 2011.
17. Bequette, B.W. and Ogunnaike, B.A., *IEEE Contr. Syst. Mag.* 21(2): 10(2001). <https://doi.org/10.1109/37.915397>.
18. Luyben, W.L., *Process Modeling, Simulation, and Control for Chemical Engineers*, 2nd edn., McGraw-Hill, New York, 1989.
19. Campos-Salazar, J.M., Santander, F., and Keim, E., *Energy Eng.* 123(6): 14(2026). <https://doi.org/10.32604/ee.2026.075407>.
20. Campos-Salazar, J.M., Lecaros, P., and Sandoval-García, R., *AIMS Electron. Electr. Eng.* 8(4): 441(2024). <https://doi.org/10.3934/electreng.2024021>.
21. Lhuillier, D., *Phys. A (Amsterdam, Neth.)* 165(3): 303(1990). [https://doi.org/10.1016/0378-4371\(90\)90002-A](https://doi.org/10.1016/0378-4371(90)90002-A).
22. Åström, K.J. and Hägglund, T., *Advanced PID Control*, The Instrumentation, Systems, and Automation Society (ISA), Research Triangle Park, NC, USA, 2006.
23. Åström, K.J. and Murray, R. M., *Feedback Systems: An Introduction for Scientists and Engineers*, Princeton University Press, Princeton, NJ, USA, 2008. <https://doi.org/10.1515/9781400828739>.
24. Edwards, C. and Spurgeon, S.K., *Sliding Mode Control: Theory and Applications*, CRC Press, London, 1998.
25. Young, K.D., Utkin, V.I., and Ozguner, U., *IEEE Trans. Control Syst. Technol.* 7(3): 328(1999). <https://doi.org/10.1109/87.761053>.
26. Khalil, H.K., *Nonlinear Control*, 1st edn., Pearson, Boston, USA, 2014.
27. Slotine, J.-J.E. and Li, W., *Applied Nonlinear Control*, Prentice Hall, Englewood Cliffs, NJ, USA, 1991.

28. Yu, S., Yu, X., Shirinzadeh, B., et al., *Automatica* 41(11): 1957(2005). <https://doi.org/10.1016/j.automatica.2005.07.001>.
29. Heydari, E., Varjani, A.Y., and Diallo, D., *Control Eng. Pract.* 104: 104635(2020). <https://doi.org/10.1016/j.conengprac.2020.104635>.
30. Levant, A., *Int. J. Control* 76(9-10): 924(2003). <https://doi.org/10.1080/0020717031000099029>.
31. Liu, J., *Sliding Mode Control Using MATLAB*, Academic Press, London, 2017.
32. Rahman, M., Avelin, A., and Kyprianidis, K., *Processes* 8(10): 1231(2020). <https://doi.org/10.3390/pr8101231>.
33. Jung, J., Choi, H.-K., Son, S.H., et al., *Comput. Chem. Eng.* 158: 107640(2022). <https://doi.org/10.1016/j.compchemeng.2021.107640>.
34. Lobo, L.A. and Bolton, T.S., *TAPPI J.* 12(2): 69(2013). <https://doi.org/10.32964/TJ12.2.69>.
35. Ventura, C., Blanco, A., Negro, C., et al., *TAPPI J.* 6(7): 17(2007). <https://doi.org/10.32964/TJ6.7.17>.
36. Duffy, G.G., Kazi, S.N., and Chen, X.D., *Appl. Therm. Eng.* 31(14-15): 2971(2011). <https://doi.org/10.1016/j.applthermaleng.2011.05.028>.
37. García, C.E., Prett, D.M., and Morari, M., *Automatica* 25(3): 335(1989). [https://doi.org/10.1016/0005-1098\(89\)90002-2](https://doi.org/10.1016/0005-1098(89)90002-2).
38. Correa, I.B., de Souza, M.B., and Secchi, A.R., *Chem. Eng. Res. Des.* 207: 505(2024). <https://doi.org/10.1016/j.cherd.2024.06.026>.
39. Bhartiya, S., Dufour, P., and Doyle, F.J. III, *AIChE J.* 49(2): 411(2003). <https://doi.org/10.1002/aic.690490212>.
40. Kammer, L.C., Ein-Mozaffari, F., Dumont, G.A., et al., *J. Process Control* 15(1): 31(2005). <https://doi.org/10.1016/j.jprocont.2004.04.008>.

ABOUT THIS PAPER

Cite this article as:

Campos-Salazar, J.M., *TAPPI J.* 25(9): 523(2026). <https://doi.org/10.32964/TJ25.9.523>

DOI: <https://doi.org/10.32964/TJ25.9.523>

ISSN: 0734-1415

Publisher: TAPPI Press

Copyright: ©TAPPI Press 2026

[About this journal](#)

41. Paterson, D.T., Eaves, T.S., Hewitt, D.R., et al., *AIChE J.* 67(9): e17277(2021). <https://doi.org/10.1002/aic.17277>
42. TAPPI TIP 0410-14, "Generalized method for determining the pipe friction loss of flowing pulp suspensions," TAPPI Press, Peachtree Corners, GA, USA, 2021.
43. Ja, A., Benjelloun, S., Ghidaglia, J.M., et al., *Int. J. Multiphase Flow* 181: 105011(2024). <https://doi.org/10.1016/j.ijmultiphaseflow.2024.105011>.
44. Gruszczyński, M.F., Kostecki, S., Zieliński, S., et al., *Sensors* 22(20): 7857(2022). <https://doi.org/10.3390/s22207857>.
45. Andruschak, S.V., Besedin, P.V., and Doroganov, E.A., *IOP Conf. Ser.: Mater. Sci. Eng.* 560(1): 012139(2019). <https://doi.org/10.1088/1757-899X/560/1/012139>.
46. Shi, B. and Wei, J., *Adv. Mech. Eng.* 6: 678271(2014). <https://doi.org/10.1155/2014/678271>.

ABOUT THE AUTHORS

I chose this topic for research because batch digester blowdown is a little-explored, highly nonlinear transient process where pulp consistency, hydraulic resistance, channeling and drainability evolve and directly affect discharge behavior. Previous studies have addressed pulp dewatering, channeling, rheology, and hydraulic transport separately. My research integrates these effects into a unified dynamic model and additionally compares fixed-gain PI and nonlinear SMC for blowdown flow control.

The main challenge in this study was preserving physical consistency while keeping the model suitable for control design. I addressed this using coupled mass balances, algebraically reconstructed slurry properties, and a reduced nonlinear momentum-based hydraulic model.

Through this research, I found that variations in consistency, density, level, and hydraulic resistance modify the operating point throughout blowdown, making fixed-point linear control increasingly restrictive. The most interesting result was that SMC substantially reduced tracking errors compared with PI while producing almost the same cumulative electrical energy and batch-specific SEC; its main benefit

was therefore robustness rather than energy reduction.

Mills could use the framework presented here for controller design, virtual disturbance testing, pump head assessment, and energy analysis before plant implementation. After industrial parameter identification, it could also support digital-twin or operator-support applications.

The next step in this research is experimental identification and plant-scale validation using actual digester data, followed by incorporation of measured pump efficiency maps and testing of the controller through hardware-in-the-loop or industrial trials.



Campos-Salazar

Campos-Salazar is guest researcher for the Vice-Rectorate for Research, Innovation and Creation at the University of Los Lagos in Osorno, Los Lagos, Chile. Email Campos-Salazar at jose.campossalazar@ulagos.cl.

PROCESS CONTROL

47. Wilson, K.C., Clift, R., and Sellgren, A., *Powder Technol.* 123(1): 19(2002). [https://doi.org/10.1016/S0032-5910\(01\)00423-5](https://doi.org/10.1016/S0032-5910(01)00423-5).
48. Sochi, T., *J. Pet. Sci. Eng.* 78(3-4): 582(2011). <https://doi.org/10.1016/j.petrol.2011.08.006>.
49. Metzner, A.B. and Reed, J.C., *AIChE J.* 1(4): 434(1955). <https://doi.org/10.1002/aic.690010409>.
50. Nault, J.D. and Karney, B.W., *J. Hydraul. Eng.* 142(9): 04016025(2016). [https://doi.org/10.1061/\(ASCE\)HY.1943-7900.0001145](https://doi.org/10.1061/(ASCE)HY.1943-7900.0001145).
51. Rundo, M., Casoli, P., and Lettini, A., *Sustainability* 13(13): 7174(2021). <https://doi.org/10.3390/su13137174>.
52. Faghani, A., Sen, S., Vaezi, M., et al., *Biosyst. Eng.* 200: 284(2020). <https://doi.org/10.1016/j.biosystemseng.2020.10.009>.
53. Shibani, B., Ambure, P., Purohit, A., et al., *Comput. Chem. Eng.* 170: 108138(2023). <https://doi.org/10.1016/j.compchemeng.2023.108138>.
54. Wu, J., Graham, L., Wang, S., et al., *Miner. Eng.* 23(9): 705(2010). <https://doi.org/10.1016/j.mineng.2010.04.008>.
55. Anton, A.-A., Coccoceanu, A., and Muntean, S., *Appl. Sci.* 12(22): 11450(2022). <https://doi.org/10.3390/app122211450>.
56. Hashemi, S.A., Wilson, K.C., and Sanders, R.S., *Powder Technol.* 262: 183(2014). <https://doi.org/10.1016/j.powtec.2014.04.021>.
57. Gevorkov, L., Domínguez-García, J.L., Rassölkin, A., et al., *Energies* 15(11): 4068(2022). <https://doi.org/10.3390/en15114068>.
58. Lavrič, H., Drobnič, K., and Fišer, R., *Appl. Sci.* 14(22): 10430(2024). <https://doi.org/10.3390/app142210430>.
59. Li, M., He, Y., Jiang, R., et al., *Particuology* 66: 38(2022). <https://doi.org/10.1016/j.partic.2021.08.004>.
60. Wang, Y., Zhang, H., Han, Z., et al., *Processes* 9(9): 1538(2021). <https://doi.org/10.3390/pr9091538>.
61. Yu, X., Yang, X., and Zhang, J., *Renewable Energy* 133: 1426(2019). <https://doi.org/10.1016/j.renene.2018.08.100>.
62. Skogestad, S., *J. Process Control* 13(4): 291(2003). [https://doi.org/10.1016/S0959-1524\(02\)00062-8](https://doi.org/10.1016/S0959-1524(02)00062-8).
63. Smith, C.A. and Corripio, A.B., *Principles and Practice of Automatic Process Control*, 2nd edn., Wiley, New York, 1997.
64. Ogata, K., *Modern Control Engineering*, 5th edn., Pearson, Upper Saddle River, NJ, USA, 2010.
65. Wang, Q.-G., Zhang, Z., Åström, K.J., et al., *J. Process Control* 19(2): 349(2009). <https://doi.org/10.1016/j.jprocont.2008.04.012>.
66. Åström, K.J. and Rundqwist, L., *1989 American Control Conference*, IEEE, Piscataway, NJ, USA, 1989, p. 1693. <https://doi.org/10.23919/ACC.1989.4790464>.
67. Slotine, J.-J. E. and Sastry, S.S., *Int. J. Control* 38(2): 465(1983). <https://doi.org/10.1080/00207178308933088>.
68. Guan, C. and Pan, S., *Control Eng. Pract.* 16(11): 1275(2008). <https://doi.org/10.1016/j.conengprac.2008.02.002>.
69. Slotine, J.-J.E., *Int. J. Control* 40(2): 421(1984). <https://doi.org/10.1080/00207178408933284>.
70. Gibson, I.H., *ISA Trans.* 33(2): 165(1994). [https://doi.org/10.1016/0019-0578\(94\)90049-3](https://doi.org/10.1016/0019-0578(94)90049-3).



TAPPI OnDemand is a centralized platform that brings trusted industry knowledge together in one place to support continuous learning, informed decision-making, and professional growth



Webinars

Expert-led sessions on real-world industry challenges.



Podcasts

Industry conversations on trends and innovation.



Articles

Concise insights to keep you informed.



Education

Structured learning to build skills and expertise.

Start exploring today at **ondemand.tappi.org**