Principles of hydrodynamic instability: applications in coating systems

Part 3: A generalized view of instability and bifurcation

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ABSTRACT *With new high-speed paper machines, coating rheology and flow instability are critical issues. The ideal flow in a coating device is steady state and two-dimensional. In other words, it would have no velocity components or momentum transfers in the cross-machine direction. In coating, complex fluids are applied to paper at high speeds under extreme flow conditions, so the problems of fluid mechanics and instability are often unique, requiring fundamental, in-depth analysis.*

Flow-through coating devices as nonlinear dynamic systems can only behave in a specific and consistent manner. By knowing the fundamental physical and dynamical principles (such as general behavior and transitions between steady and time-periodic states), we are better equipped to deduce and interpret information from experiments or computational analysis of instability problems. Mathematical examples are included to provide simple but practical working tools.

KEYWORDS
Blade coaters
Coating
Fluid dynamics
Fluid mechanics
Hydrodynamic stability
Rheology
Viscous flow

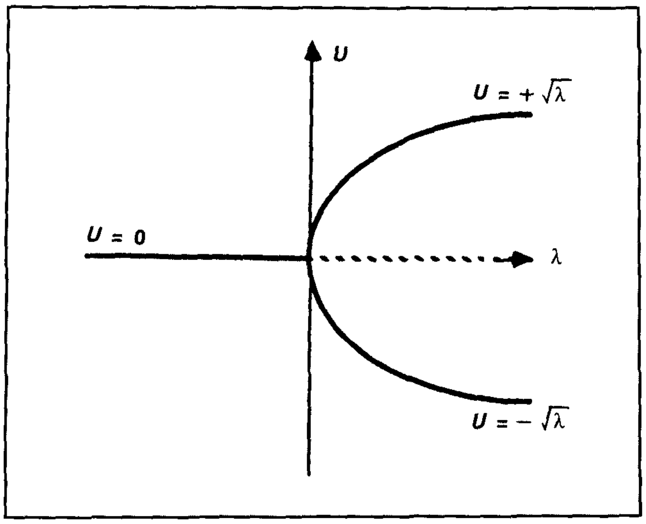
"Science is nothing without generalizations. Detached and ill-assorted facts are only raw material, and in the absence of a theoretical solvent, have but little nutritive value." —Lord Rayleigh (1884)

Coating flows have complex (nonlinear) mechanical and dynamical characteristics. To apply coating on paper demands, by nature, flow through complicated coating devices from regions of relatively low shear to very high shear. At the same time, a stable steady two-dimensional stream is often required at high speeds, and deviations from this situation may not be tolerated. It is these severe restrictions that demand a better understanding of the dynamics and stability of the flow in coating devices. Under identical operating conditions, the flow in a short-dwell coater may be steady and nearly two-dimensional, steady and three-dimensional, or unsteady and three-dimensional. The conventional modeling methods and the classical approaches cannot fully explain and predict the physics of these problems.

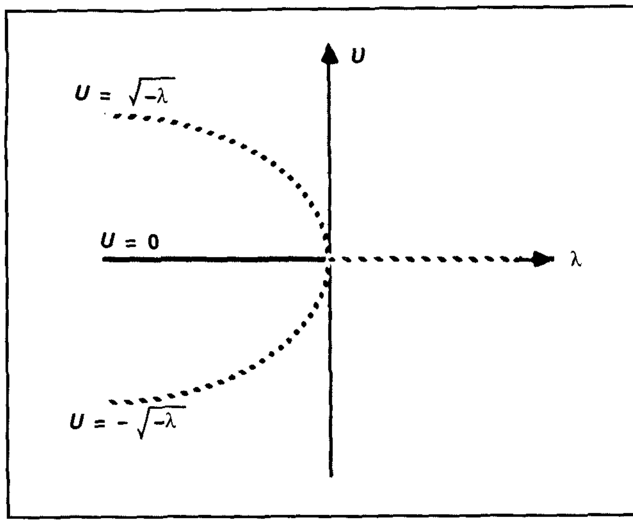
An explanation and generalization of the possible behaviors of the coating flows as dynamical systems is possible through simple nonlinear models and logical reasoning. This approach will provide the fluid mechanician and the engineer with a frame of reference and guidelines for cost-effective problem solving and improved systems designing. Bifurcation theory provides some of the tools that are required for this task. Results from simple experiments, such as flow examinations, can be used in conjunction with some fundamental principles of bifurcation theory to outline and predict useful features of the system under study without actually performing extensive and costly experiments. A vivid example of this approach is Benjamin's analysis of bifurcation in steady flow of a viscous fluid flow in a circular Couette system (*1*).

Here, we review some relevant and basic principles in bifurcation (one parameter) of equilibrium and periodic states of dynamical systems. The concepts are illustrated with simple mathematical models. The subject matter is simple, yet it can be of enormous use when dealing with

1. A supercritical (pitchfork) bifurcation (one-sided)



2. A subcritical (pitchfork) bifurcation (one-sided)



complex dynamical systems such as coating devices.

Equilibrium states

When two equilibrium solutions of a system of equations with distinct tangents intersect one another, the point of intersection is a singular point of bifurcation. In mathematical terms, let the system of nonlinear evolution equations be represented as

$$\dot{\underline{x}} = \underline{F}(\underline{x},\lambda) \qquad (1)$$

where a dot over the dependent variables, $\underline{x}$, represents differentiation with respect to time, and an underbar represents an n dimensional vector. Here λ is a parameter in the system, and the analysis in this section is limited to a system of equations with only one parameter.

The equilibrium solutions of Eq. 1 can be obtained from the system of equations given by

$$\underline{F}(\underline{x}(\lambda),\lambda) = \underline{0} \qquad (2)$$

The dependent variables, $\underline{x}$, are functions of the parameter, λ. A regular point $(\underline{x}_r,\lambda_r)$ is a point where the Jacobian, $\underline{F}_{\underline{x}}(\underline{x}_r,\lambda_r)$, and $\underline{F}_\lambda(\underline{x}_r,\lambda_r)$ are not zero. The implicit function theorem (*2*) guarantees that a unique solution curve passes through a regular point. Bifurcation theory deals basically with the study of the behavior of equilibrium solutions when the implicit function theorem breaks down. At the singular point $(\underline{x}_s,\lambda_r)$ where the determinant of $\underline{F}_{\underline{x}}(\underline{x}_s,\lambda_s)$ vanishes and $\underline{F}_\lambda(\underline{x}_s,\lambda_s) = 0$, the implicit function theorem is no longer valid.

A bifurcation point is a singular point at which two solutions of Eq. 2 intersect with a distinct tangent (double point). A simple example is the point $U = 0$, $R = 5$ in Fig. A1 of the appendix where two solutions, $U = 0$ and $U = R - 5$, intersect. As a second example, consider the following evolution equation:

$$\dot{U} = U(\lambda - U^2) \qquad (3)$$

The two equilibrium solutions, $U = 0$ and $U = \pm\lambda^{1/2}$, intersect at the origin. At the bifurcation point, the null solution, $U = 0$, loses stability and gives rise to a different stable solution. In other words, an exchange of stability takes place at the origin. The bifurcation diagram is shown in **Fig. 1**. When the bifurcating solution lies on the unstable (stable) side of the trivial solution, then it is termed "supercritical" or ("subcritical").

An example of a subcritical bifurcation is given by $\dot{U} = U(\lambda + U^2)$ as shown in **Fig. 2**. In contrast to the previous example, Eq. 3, here there is no exchange of stability at the origin, only a local loss of stability of the trivial solution.

A "transcritical" bifurcation occurs when the stable (unstable) section of the bifurcating nontrivial solution is on the unstable (stable) side of the bifurcation point of the trivial solution. Figure A1 in the appendix is an example of a transcritical bifurcation. The following theorem summarizes the main results of a simple bifurcation point.

Theorem:

(1) Let $\underline{x} = \underline{0}$ be the trivial solution of $\underline{F}(\underline{x},\lambda) = \underline{0}$ for all λ.

(2) Let $\sigma(\lambda)$ be a simple eigenvalue of $\underline{F}_{\underline{x}}(\underline{0},\lambda)$.

(3) At the critical parameter, λ_c, let the eigenvalue $\sigma(\lambda_c) = 0$, and let $(d\sigma/d\lambda)_{\lambda=\lambda_c} \neq 0$

Then:

(1) There is one and only one solution branch that intersects the trivial solution at λ_c.

(2) The critical mode of a supercritical bifurcating solution is stable.

(3) The critical mode of a subcritical bifurcating solution is unstable.

Transition from steady to time-periodic states

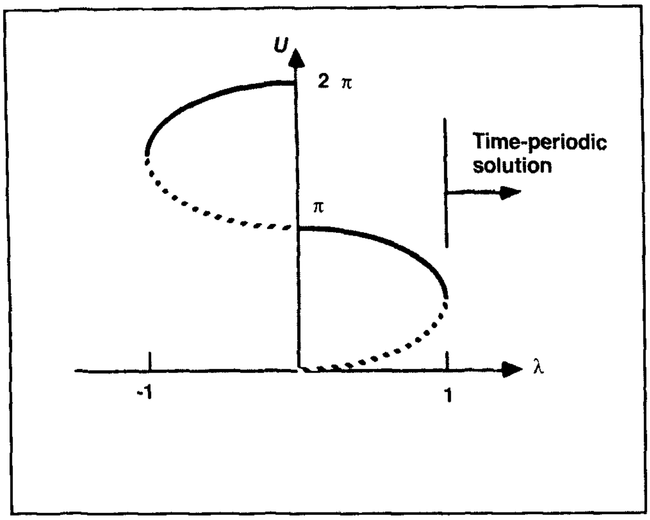
A transition to a time-periodic state can occur by at least two distinct mechanisms. In some dynamic systems, the steady state solution ceases to exist beyond the onset of time-periodicity where only an oscillatory solution can exist. In this case, the transition occurs through a turning point in the steady state solution branch. An example of this transition is given by the simple model

$$\dot{U} = \sin U - \lambda \qquad (4)$$

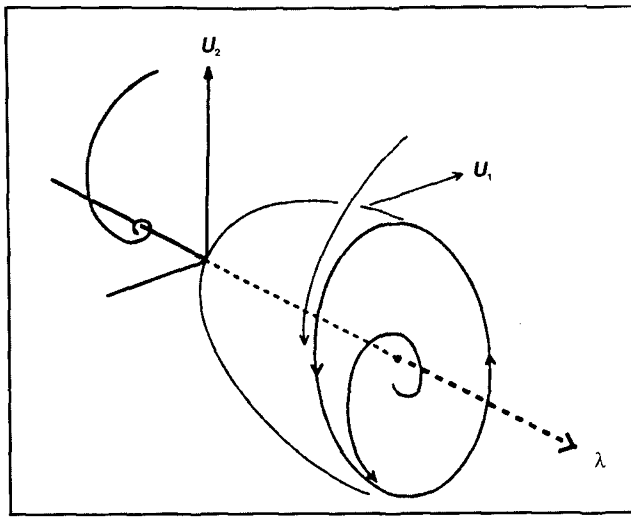
where $0 < U < 2\pi$

Two steady state solutions exist for $0 < \lambda < 1$. As shown in **Fig. 3**, the stable and unstable equilibrium solutions meet at a turning point at $\lambda = 1$. Beyond this point for $\lambda > 1$, there are no steady solutions, only a time-

3. Transition from steady to time-periodic state through a turning point



4. Generation of stable limit cycles through a supercritical Hopf bifurcation



periodic state for which frequency increases from zero at the transition point.

The second mechanism for transition from steady to time-periodic state involves a loss of local stability of the steady state solution and exchange of stability to a time-periodic branch. The point where an equilibrium branch destabilizes and gives rise to a time-periodic branch is a Hopf bifurcation point.

Let us consider a simple example of a Hopf bifurcation. Consider the system of evolution equations:

$$\dot{U}_1 = -U_2+U_1(\lambda - U_1^2 - U_2^2) \qquad (5A)$$

$$\dot{U}_1 = U_2+U_1(\lambda - U_1^2 - U_2^2) \qquad (5B)$$

From the form of the equations, it is obvious that the trivial solution, $U_1 = U_2 = 0$, is the only equilibrium solution for all λ. Writing Eq. 5 in local form, we can easily show that this solution destabilizes at $\lambda = 0$ and remains unstable for $\lambda > 0$. Although there is a loss of stability at $\lambda = 0$, there is no turning point and no exchange of stability within the space of equilibria (no stationary bifurcation). The Jacobian has a pair of complex conjugate eigenvalues that cross the imaginary axis (with a nonzero speed) when $\lambda = 0$. To study this transition, let us reformulate the system represented in Eqs. 5A and 5B in polar coordinates (R,θ) defined as $R\cos\theta = U_1$ and $R\sin\theta = U_2$. After some manipulations, Eq. 5 reduces to

$$\dot{R} = R(\lambda - R^2) \qquad (6A)$$

$$\dot{\theta} = 1 \qquad (6B)$$

For $\lambda \geq 0$, the solution $R = \lambda^{1/2}$ results in a time-periodic orbit, a *limit cycle*, given by $\theta = t$ and $R = \lambda^{1/2}$. Therefore, the orbit has a period 2π, and its amplitude (radius), as shown in **Fig. 4**, grows with $\lambda^{1/2}$. Since

$$\dot{R} < 0 \text{ for } R > \lambda$$

and

$$\dot{R} > 0 \text{ for } R < \lambda$$

the time-periodic state is stable; that is, all trajectories in the phase plane approach the limit cycle. The two branches of the equilibrium solution represent stable focus ($\lambda < 0$) and unstable focus ($\lambda > 0$) points. At $\lambda = 0$, a periodic solution bifurcates from the equilibrium branch, and an exchange of stability from steady to time-periodic state takes place.

In general, a Hopf bifurcation involves the crossing of the imaginary axis on the complex plane by a pair of complex conjugate eigenvalues of the Jacobian. The following theorem generalizing this transition was proven by E. Hopf (*3*).

Theorem:

(1) Let $\underline{x}_0$ be the equilibrium solution of the nonlinear system of equations $\underline{F}(\underline{x},\lambda) = 0$ which destabilizes at $\lambda = \lambda_c$.

(2) The Jacobian $F_x(x_0,\lambda_0)$ has a simple pair of purely imaginary eigenvalues, $\mu(\lambda_0) = \pm i\beta$, and no other eigenvalues with zero real parts.

(3) $[d\mu(\lambda)/d\lambda]_{\lambda=\lambda_0} \neq 0$; that is, the crossing has nonzero speed.

Then:

(1) There is a bifurcation at $(\underline{x}_0,\lambda_0)$ to a unique (after fixing the phase) time-periodic solution branch.

(2) The period of the limit cycle approaches $2\pi/\beta$ as λ approaches λ_0.

(3) The bifurcating periodic branch is one-sided. (Transcritical bifurcations to periodic solutions are not possible.) The critical mode of a supercritical bifurcating solution is stable, and the critical mode of a subcritical bifurcating solution is unstable.

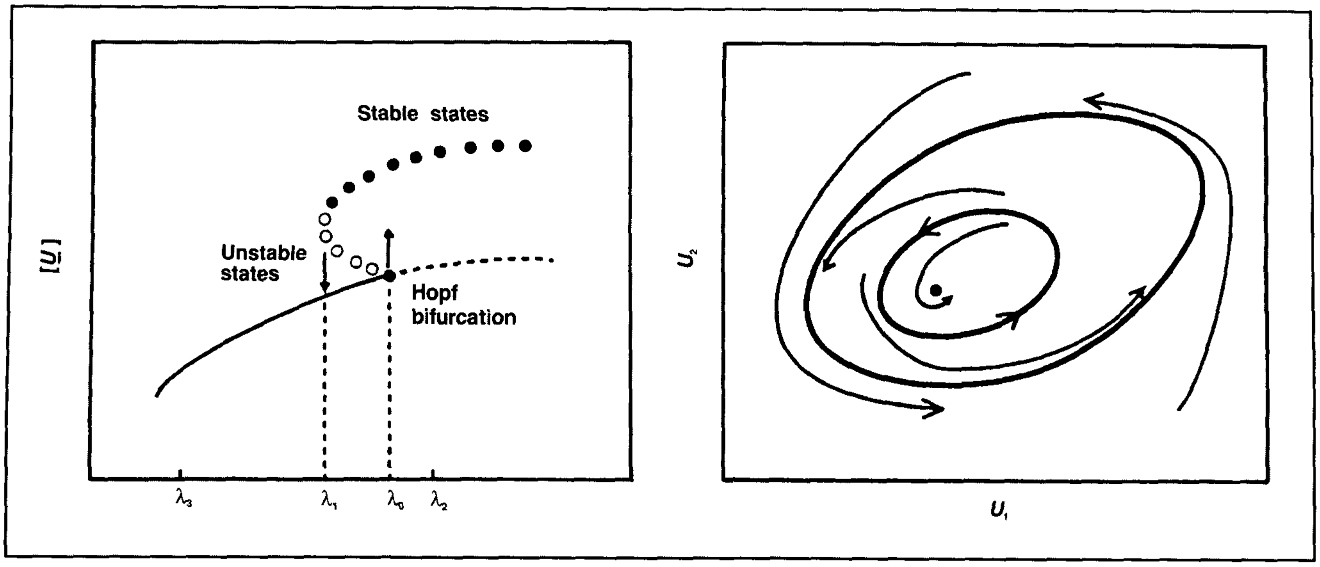
(4) Hopf bifurcation is generic; that is, small perturbations of the system of equations will not change the result qualitatively.

The exceptional bifurcation where multiple eigenvalues cross the imaginary axis or where transversality does not hold are discussed by Iooss and Joseph (*4*).

This example, Eq. 5, illustrates a supercritical Hopf bifurcation where a soft loss of stability or soft generation of limit cycles occurs. In contrast, in the case of a subcritical Hopf bifurcation, the periodic branch is initially unstable and turns back over the stable equilibrium branch (**Fig. 5**). It then gains stability through a turning point at $\lambda = \lambda_1$. Between λ_1 and λ_0, there are three solutions: a stable equilibrium, a stable time-periodic, and an unstable time-periodic.

As mentioned, locally unstable states are not observable in real experiments and can be obtained only through computational analysis. A subcritical bifurcation with a subsequent turning point and transition to a stable state generates a bistable

5. Generation of stable and unstable limit cycles through a subcritical Hopf bifurcation. Left: bifurcation diagram where (U) is some scalar measure of the solution. The circles represent stable and unstable time-periodic state. Right: stable and unstable limit cycles and the locally stable equilibrium point on the phase plane for $\lambda > \lambda_0$.



situation with hysteresis, which is of considerable practical importance. Consider an experiment which is marching along the equilibrium branch of Fig. 5 with gradually increasing control parameter λ. (The control parameter could be the machine speed, the Reynolds number, or any other parameter in the experiment which has primary influence on the dynamics.) As λ approaches λ_0, there will be a sudden transition from steady state to a time-periodic state. In other words, at $\lambda = \lambda_0$, a hard loss of stability to a finite-amplitude time-periodic motion takes place. Now if the experiment marches backward on the periodic branch from $\lambda = \lambda_1$, it passes λ_0 without transitions until it reaches $\lambda = \lambda_1$. At this point, the system jumps from the time-periodic state to a steady state and continues on the equilibrium branch as λ decreases.

Dynamical behavior at the onset of macroscopic coating weight nonuniformities (streaks)

To demonstrate how the above information can be used to understand and explain the dynamics of coating systems, let us consider the problem of nonuniform coating weight distribution with short dwell coaters.

Pilot trials show that within a specific range of machine speed and solids concentration, the onset of streaks involves an intermittent behavior (not to be confused with the intermittency route to unsteady nonperiodic states). The coated surface shows patches of streaks followed by a relatively uniform coated region. What is causing this behavior?

Consider a fluid flow experiment that exhibits the same dynamics as in Fig. 5, and which, due to some external effects (not necessarily flow dependent), is forced to oscillate between λ_3 and λ_2. Since a bistable situation exists in the subinterval λ_0 to λ_1 in combination with the hysteresis effect generated by the subcritical Hopf bifurcation, a bursting phenomenon will be observed in the experiment. During the pass from left to right along the equilibrium branch as λ increases from λ_3 to λ_0, the flow is stationary (steady) and uniform. As the control parameter passes λ_0, the system jumps to the periodic branch. This generates initial transients, which could involve several large amplitude oscillations that could persist until λ approaches λ_3. The alternatively active and stationary modes of the flow result in periodic bursts. This dynamical behavior has also been observed in other experimental and industrial applications (*5*).

Transition from the primary steady-state to a time-periodic state with small oscillation amplitude has been observed in the flow of Newtonian fluids in a lid-driven cavity simulating the pond of a short-dwell coater. Although not rigorously proven yet, this transition is most likely to be through a Hopf bifurcation. Considering that the additional force introduced by the elasticity of the fluid makes a subcritical Hopf bifurcation possible, the mechanism proposed becomes viable.

Here, the whole coating system is considered as a dynamical unit with multiple parameters. The parameter that may drive this unit to oscillate through the critical region, λ_3 to λ_2, could be, for example, the periodic shear-induced pigment structures (bridges) that may form under the blade, the hydrodynamic effects upstream of the blade, or even some mechanical property of the system.

By pinpointing the mechanism and the parameters that destabilize and control the coating system, we can design a more stable, reliable, and cost-effective device.

Literature cited

1. Benjamin, T. B., Proc. Royal Soc. A 359: 1(1978).
2. Iooss, G. and Joseph, D. D., Elementary stability and bifurcation theory, Springer-Verlag, New York, 1980.
3. Hopf, E., Abzweigung einer periodischen Lösung von einer stationären Lösung eines Differential systems, Ber. Verh. Sächs. Akad. Wiss. Leipzig, 1942, Math.-phys. Kl. 94, 1942, pp. 1–22; translated in *The Hopf Bifurcation and its application*, (J.E. Marsden and M. McCracken, Eds.), New York: Springer-Verlag, 1976, p. 163.
4. Iooss, G. and Joseph, D. D., Elementary

stability and bifurcation theory, Springer-Verlag, New York, 1980.
5. Rinzel, J. and Troy, W. C., J. Chem. Phys. 76: 1775(1982).

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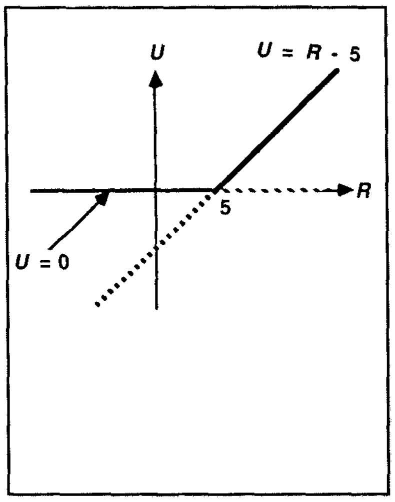
A1. Stability characteristics of the solutions $U = 0$ and $U = R - 5$



Appendix

The conservation principles in fluid mechanics can be written in the form of a system of differential equations that govern the basic motion of the fluid. As they stand, these equations cannot predict the stability properties of the base flow. To examine stability, a disturbance can be imposed on the base flow, and the resulting equations will show if the disturbance grows (unstable) or decays (stable).

For illustration purposes, let us assume that Eq. A1 governs a system with dependent variable U:

$$dU/dt = -U(U - R + 5) \qquad \text{(A1)}$$

R is a real number that serves as the parameter in the problem. The steady state solutions to this equation can be found by setting $dU/dt = 0$ and solving the algebraic equation, Eq. A2:

$$U(U - R + 5) = 0 \qquad \text{(A2)}$$

Therefore, $U = 0$ and $U = R - 5$ are each equilibrium solutions of the evolution Eq. A1. But which solution is stable and can physically exist? To answer this question, we examine the stability of each solution by analyzing the behavior of that solution with respect to an infinitesimal disturbance.

More specifically, we study the behavior of $U + \epsilon u(t)$, where $\epsilon u(t)$ is an infinitesimal disturbance. Substituting the disturbed solution into Eq. A1, it is easy to show that the disturbance, $u(t)$, imposed on the base solution, U, behaves as

$$u(t) = \exp(s\,t)$$

where the growth parameter s is given by

$$s = -[2U - R - 5)] \qquad \text{(A3)}$$

Depending on the magnitude of R, the parameter s could be positive or negative and could result in unstable or stable solutions, respectively.

For the base solution, $U = R - 5$, the disturbance $u(t)$ grows when $R < 5$ and decays when $R > 5$. This solution is therefore unstable when $R < 5$, and it stabilizes at the critical value, $R_c = 5$.

The null solution, $U = 0$, is stable when $R < 5$, it destabilizes at $R = 5$, and it remains unstable for $R > 5$.

These results are summarized in **Fig. A1**, where stable and unstable solution branches are shown with solid and broken lines, respectively. This analysis shows that for $R < 5$, the only physically possible state is $U = 0$, and for $R > 5$, $U = R - 5$ is the only stable state.