Principles of hydrodynamic instability: application in coating systems

Part 1: Background

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ABSTRACT *With new high-speed paper machines, coating rheology and flow instability are critical issues. The ideal flow in a coating device is steady state and two-dimensional. In other words, it would have no velocity components or momentum transfers in the cross-machine direction. In coating, complex fluids are applied to paper at high speeds under extreme flow conditions, so the problems of fluid mechanics and instability are often unique, requiring fundamental, in-depth analysis.*

Flow through coating devices as nonlinear dynamic systems can only behave in a specific and consistent manner. By knowing the fundamental physical and dynamical principles (such as general behavior and transitions between steady and time-periodic states), we are better equipped to deduce and interpret information from experiments or computational analysis of instability problems. Mathematical examples are included to provide simple but practical working tools.

KEYWORDS
Blade coaters
Coating
Fluid dynamics
Fluid mechanics
Hydrodynamic stability
Rheology
Viscous flow

> Yet not every solution of the equations of motion, even if it is exact, can actually occur in Nature. The flows that occur in Nature must not only obey the equations (conservation principles) of fluid dynamics but also be stable (*1*).

Hydrodynamic instability has been the central focus in the study of the physics of fluid motion for more than a century. The reason this issue persists as one of the dominating ones in the field of continuum mechanics is that a flow system can satisfy all of the principles of fluid mechanics conservation and yet be unstable and therefore nonexistent in nature.

In general, every laminar flow pattern will become unstable at some critical flow parameter and will undergo a transition to a qualitatively different flow. The new flow system could be steady, periodic, or unsteady. Here, the key issue is transition between qualitatively different systems or states.

The purpose of this study is to outline some techniques that are available for analyzing and predicting flow instability. Through understanding the fundamental physical and dynamical principles, we are better equipped to interpret analyses of instability problems. We will examine the principles of hydrodynamic instability in coating systems in three parts. Part 1 will outline the general principles of flow instability, Part 2 will describe flow instability in a number of coating systems, and Part 3 will present an overview of coating flows as nonlinear hydrodynamic systems.

Background on flow instability

The study of flow instability started in the nineteenth century with physicists such as Helmholtz, Kelvin, Rayleigh, and Reynolds. The classical experiments of Osborne Reynolds (*2*) on the flow in a pipe is perhaps the most well-known study in fluid mechanics. In these experiments, as illustrated in **Fig. 1**, water was drawn out of a large glass tank through a circular tube while a streak of highly colored water entered simultaneously with the clear water. The results of these experiments in Reynolds's own words are as follows (*2*):

1. When the velocities were sufficiently low, the streak of colour extended a beautiful straight line through the tube. . . .
2. If the water in the tank had not quite settled to rest, at sufficiently low velocities, the streak would shift about the tube, but there was no appearance of sinuosity.
3. As the velocity was increased by small stages, at some point in the tube, . . . the colour band would all at once mix up with the surrounding water, and fill the rest of the tube with a mass of coloured water, as in Fig. 1B. . . . On viewing the tube by a light of an electric spark, the mass of colour resolved itself into a mass of more or less distinct curls, showing eddies, as in Fig. 1C. . . . the critical velocity was very sensitive to disturbance in the water before entering the tube. . . . This at once suggested the idea that the condition might be one of instability for disturbances of certain magnitude and stable for smaller disturbances.

Reynolds repeated these experiments with tubes of different diameters and established a dimensionless parameter, VD/ν, based on velocity V, diameter D, and kinematic viscosity ν. This parameter, which is now referred to as the "Reynolds number," defines the class of dynamically similar flows in a pipe and many other flow systems.

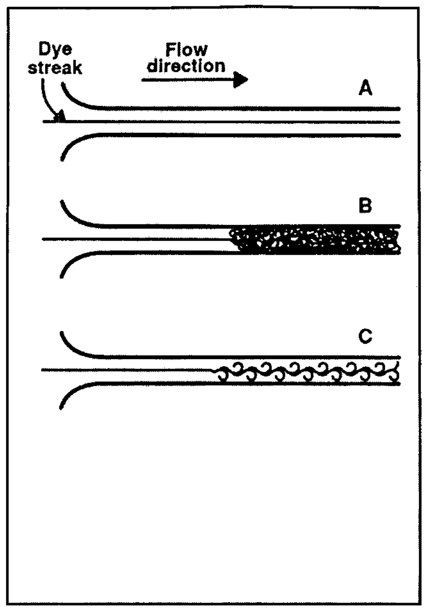
Defining the critical parameters

One step in the study of fluid dynamics systems is to define the important parameters in the problem. These parameters could be the Reynolds number or any dimensionless combination of length, velocity, and time scales with fluid properties. The stability analysis involves pinpointing the critical value of these parameters for the transition.

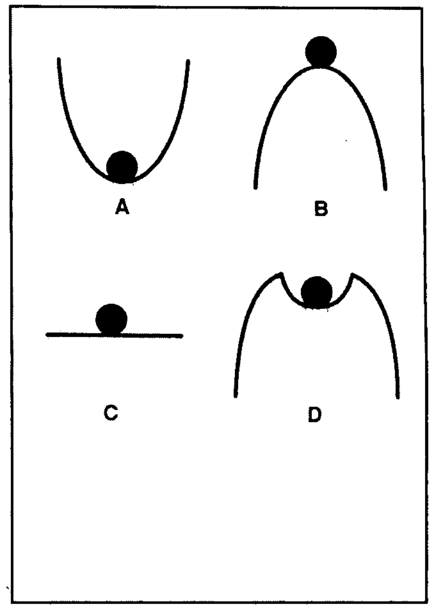
The central question in stability analysis is, can a given flow field withstand a disturbance and still return to its original state? If so, it is stable; otherwise, that particular flow pattern is unstable to the imposed disturbance. If a flow is unstable to infinitesimal disturbances, then it cannot exist in nature even if it satisfies the conservation principles.

As an example, consider Reynolds's experiment with flow in a circular tube. A one-dimensional flow with parabolic velocity profile in the tube satisfies the mass, momentum, and energy conservation principles for $0 < Re < \infty$. However, in reality, this one-dimensional flow field ceases to exist for Re greater than a critical value.* At this critical Reynolds number, there is a transition from one-dimensional Poiseuille flow to a qualitatively different flow pattern. The mass, momentum, and energy conservation equations cannot predict this transition except through stability analysis. This analysis involves examining the growth rate of all possible disturbances imposed on the base flow field as a function of the flow parameter, Re. The smallest value of the Reynolds number where a disturbance grows is the critical value for the transition.

1. (A) Laminar flow in a pipe, (B) transition to turbulent flow in a pipe, and (C) transition to turbulent flow as seen when illuminated by a spark (2)



2. Relative stability characteristics: (A) unconditionally stable, (B) unstable, (C) neutrally stable, and (D) locally stable but globally unstable



A simple demonstration of relative stability is shown in **Fig. 2**, where a ball is positioned in various states on different structures of support. State A represents an unconditionally stable system because the ball will return to its original state no matter how severely it is disturbed. In contrast, State B is an unstable one since even an infinitesimal disturbance will take the ball permanently away from its original state. This state is also referred to as a "locally" or a "linearly" unstable system. A neutrally stable system is symbolized as a ball over a flat horizontal plane, as shown in State C, where the position remains fixed regardless of the disturbance. One should not confuse a neutrally stable state with marginal stability: in marginal stability, the system loses stability at a neighboring value of the critical parameter.

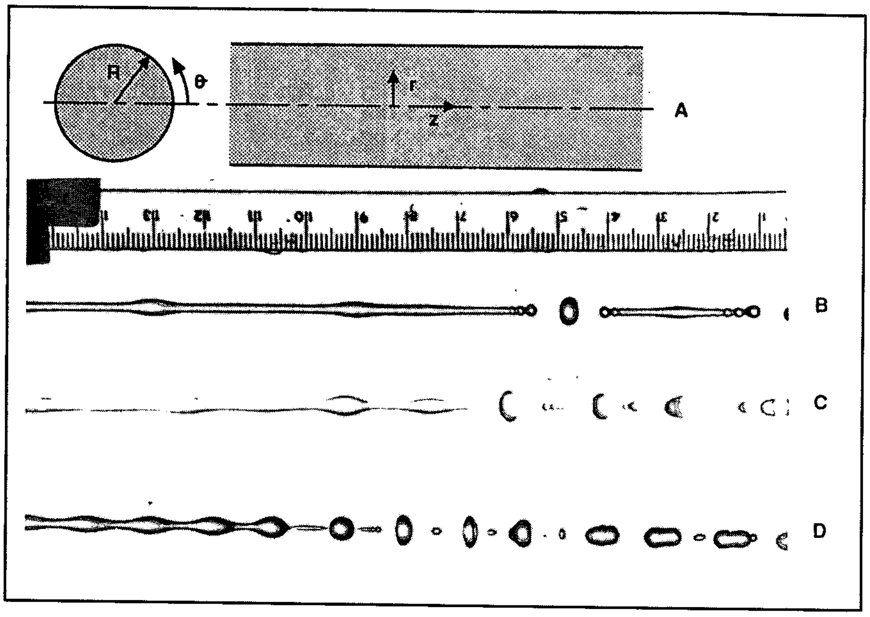
State D in Fig. 2 illustrates a more complicated situation that is physically important. The ball is in a stable state with regard to small amplitude disturbances, but it will destabilize and leave its original state when disturbed by a great enough disturbance of finite amplitude. This state is an example of a locally stable but globally unstable system.

In general, the stability analysis can be divided into two categories: (a) instability to infinitesimal disturbances (local or linear) and (b) instability to finite-amplitude disturbances (global or nonlinear). Here, we are mainly concerned with small disturbances, although globally unstable flows are also possible in coating systems (*3*).

The governing fluid dynamics equations are a set of coupled nonlinear partial differential systems that

*As Reynolds concluded from his experiments and as later experiments have supported, the Poiseuille flow in a circular tube soon becomes unstable to disturbances of finite amplitude. The critical Reynolds number for this instability is about 2000. Well-controlled laboratory experiments with extremely smooth-walled tubes have shown that the Poiseuille flow ceases to exist at a critical Reynolds number of about 40,000.

3. Break-up of a liquid jet, (A) diagram illustrating the coordinate system, where the pictures show critical modes with wavelengths (B) 84R, (C) 25R, and (D) 9.2R. Mode E is Rayleigh's prediction for the most likely destabilizing disturbance mode (6).



are well established. Closed form solutions to these equations are only available for simple ideal flows. Powerful numerical techniques, however, can be used to obtain solutions for the more complicated practical problems (4).

The general approach to analyzing the behavior of a flow to small disturbances is as follows.

Let U represent the solution for the base state for which stability is being examined. A small-amplitude disturbance, $\epsilon u'$ (where $\epsilon << 1$) is added to the base flow: $u = U + \epsilon u'(x, t)$. The resulting sum is substituted in the governing flow equations, which are nonlinear in general. The terms that involve only the base state cancel identically, and the resulting disturbance equation for u' is made linear by neglecting the higher order terms in ϵ. The critical Reynolds number where the disturbance u' starts to grow with time signals the onset of instability and the transition of the base flow U.

Let us illustrate this procedure by looking at the breakup of an inviscid liquid jet in air.

Instability of liquid jets

For illustration purposes, let us review Rayleigh's stability analysis of an inviscid cylindrical liquid jet, which is much like a thin jet of water from a hose, and of its breakup into drops (5). The steps involved in the linear stability analysis are typical and will be illustrated with this simple problem.

The equation governing the motion of an incompressible inviscid fluid is Euler's equation:

$$\partial \underset{\sim}{u}/\partial t + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} = -(1/\rho) \nabla p \qquad (1)$$

and the continuity equation

$$\nabla \cdot \underset{\sim}{u} = 0 \qquad (2)$$

where $\underset{\sim}{u}$ is the velocity vector with components u_r, u_θ, and u_z in the r, θ, and z directions, respectively (**Fig. 3**). The dependent variable $\underset{\sim}{u}$ and p are given by

$$\underset{\sim}{u} = U + \epsilon \underset{\sim}{u}' \qquad (3)$$

$$p = P + \epsilon p' \qquad (4)$$

where (U, P) correspond to the base flow, $\epsilon\ (\underset{\sim}{u}', p')$ are the disturbances, and $\epsilon << 1$. For convenience, we fix the coordinate system with the jet, so that the solution for the base flow can be written as

$$U = 0$$

$$P = P_a + \sigma/R$$

where P_a is the ambient pressure and R is the radius of the undisturbed circular jet. After the disturbance, the jet is no longer necessarily circular, and in general its shape may vary slightly from a perfect cylinder, $r = R$, to an arbitrary shape described as

$$r = \xi(\theta, z, t)$$

where the disturbance function

$$\epsilon \xi' = \xi - R$$

The dynamic and kinematic boundary conditions at the free surface of the jet ($r = \xi$) are given by

$$p = P_a + \sigma \nabla \cdot n \qquad (5)$$

$$u_r = D\xi/Dt \qquad (6)$$

where n is the outward unit vector normal to the disturbed surface, $r = \xi$. By neglecting terms that contain $0(\epsilon^2)$ and higher, the disturbance equations and corresponding boundary conditions reduce to

$$\partial \underset{\sim}{u}'/\partial t = = -(1/\rho) \Delta p' \qquad (7)$$

$$\Delta \cdot \underset{\sim}{u} = 0 \qquad (8)$$

$$p' = \frac{\sigma}{R^2}\left(\xi' + \frac{\partial^2 \xi'}{\partial \theta^2} + R^2 \frac{\partial^2 \xi'}{\partial z^2}\right) \qquad (9)$$

for $r = R$

$$u_r' = \partial \xi'/\partial t \qquad (10)$$

for $r = R$.

Since the base flow is axially symmetric and the coefficients of the disturbance equations are functions of r alone, we can write the disturbance functions in normal modes as

$$(u', p', \xi') = (\hat{u}(r), \hat{p}(r), \hat{\xi})\, e^{i\alpha z/R + st} \qquad (11)$$

where α is the wavenumber. Note that here we are considering only axisymmetric disturbances (no dependence on σ) since the jet is stable to all nonaxisymmetric modes. Our objective is to evaluate the eigenvalue, s. If $s < 0$, then the jet is stable; otherwise it is unstable.

By taking the divergence of Eq. 7 and substituting Eq. 11 into the resulting Laplacian, we get

$$\frac{d^2 p}{dr^2} + \frac{1}{r}\frac{dp}{dr} - \left(\frac{\alpha}{R}\right)^2 p = 0 \qquad (12)$$

which is just the zero-order modified Bessel equation. Considering that pressure should remain bounded as $r \to 0$, the solution is simply

$$\hat{p} = A\, I_0\,(\alpha r/R) \qquad (13)$$

and from Eq. 7,

$$\hat{\underline{u}} = -\frac{A\alpha}{\rho s R}\,[\mathrm{i}\ \mathrm{I}_0(\alpha r/R),\ \mathrm{I}'_0(\alpha r/R),\ \mathrm{o}] \qquad (14)$$

The boundary conditions (Eqs. 9 and 10) give the relation for eigenvalue, s:

$$s = \pm\left[\frac{\sigma\alpha}{\rho R^3}\,(1-\alpha^2)\,\frac{\mathrm{I}'_0(\alpha)}{\mathrm{I}_0(\alpha)}\right]^{1/2} \qquad (15)$$

Since $(\sigma\alpha)/(\rho\ R^3) \times (I_0'(\alpha)/I_0(\alpha)$ is positive for any non-zero real value of α, the eigenvalue s becomes positive when $\alpha < 1$. In other words, the jet becomes unstable to disturbances with wavelengths, $\lambda = 2\pi R/\alpha$, which are greater than the circumference, $2\pi R$, of the jet. The eigenvalue given by Eq. 15 is plotted in **Fig. 4**. The maximum growth rate corresponds to disturbance waves with wavelength $\lambda = 9R$. The idea that the dominant mode is likely to be the fastest growing mode was first suggested by Rayleigh (*5*). This is not strictly true, however, since different modes within the range $0 < \alpha < 1$ can become dominant (Fig. 3), grow, and break the jet into droplets. The size of the droplets can be obtained from the radius of the jet and the wavelength of the most destabilizing disturbance mode.

Instability of parallel shear flows

Steady two-dimensional flows with parallel streamlines occur in many coating systems. (A familiar example would be the flow under the blade in a blade coating device.) Therefore, understanding the stability properties of this class of flows is important and necessary.

The stability properties of inviscid parallel flows are well established (*7*). In general, the necessary conditions for instability are: (a) the velocity profile has a point of inflection (*8*) and (b) the absolute value of the vorticity must be a maximum at the point of inflection (*9*). Addition of viscosity could have a stabilizing or destabilizing effect. In general, the critical Reynolds number for the growth of two-dimensional disturbances in a laminar parallel flow is lower than the critical Reynolds number for growth of three-dimensional disturbances (*10*).

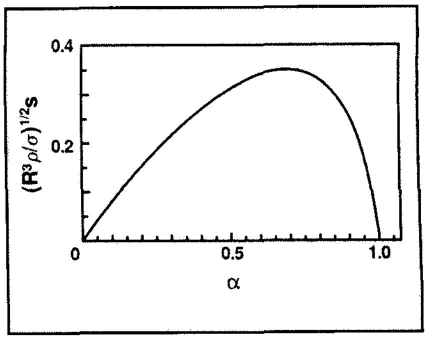
Two simple but important parallel flows are the flows between two parallel solid planes, one driven by the motion of one plane relative to the other (plane Couette flow) and the other driven by a constant pressure gradient (plane Poiseuille flow). In a blade coating system, the flow under the blade in the nearly parallel gap is roughly a combination of the plane Couette and plane Poiseuille flows. These types of flows are illustrated in **Fig. 5**.

Although the Navier-Stokes equations applied to these parallel flows reduce to linear systems and can be easily solved for the velocity profiles, solving the disturbance equations for stability analysis is not by any means trivial. In general, the disturbance equations for parallel flows reduce to a fourth-order, linear, homogeneous, ordinary differential equation with complex coefficients (*11, 12*). For a given base flow and boundary conditions, this equation, which is referred to as the Orr-Sommerfeld equation, can then be solved for the neutral stability curves.

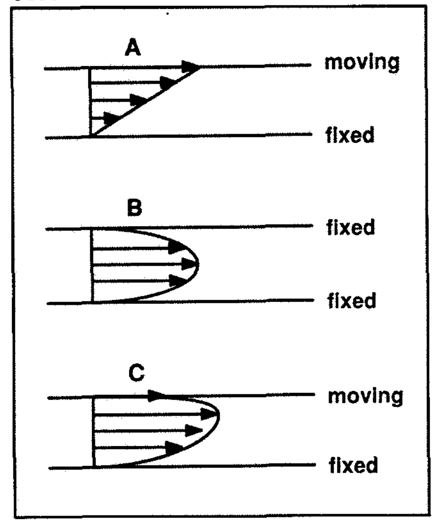
Although the solutions for the velocity profiles shown in Fig. 5 always satisfy the governing fluid mechanics conservation principles, the Orr-Sommerfeld equation shows that these parallel flows destabilize to infinitesimal disturbances at a finite critical Reynolds number. For plane Poiseuille flow, the most accurate value of the critical Reynolds number reported is R_c = 5772.22, which was obtained by Orszag (*13*) using numerical expansions of the Orr-Sommerfeld equation in Chebyshev polynomials. Therefore, the velocity profile of Fig. 5B can exist only in practice when $R < R_c$. Since the destabilizing disturbance structure is two-dimensional and symmetric with respect to the center line, when R is slightly larger than R_c, a transition occurs to a second laminar flow that is also two-dimensional, steady state, and spatially periodic in the flow direction.

In a blade coating system, if the parallel streamlines under the blade destabilize, it is likely they will be replaced initially by a steady two-dimensional laminar flow with unparallel streamlines. Only then can there be transition to a three-dimensional state.

4. Growth rates of a uniform jet with respect to unstable capillary modes



5. Flows between parallel planes, illustrating (A) Couette, (B) plane Poiseuille, and (C) Couette-Poiseuille flows



Centrifugal instability of rotating flows

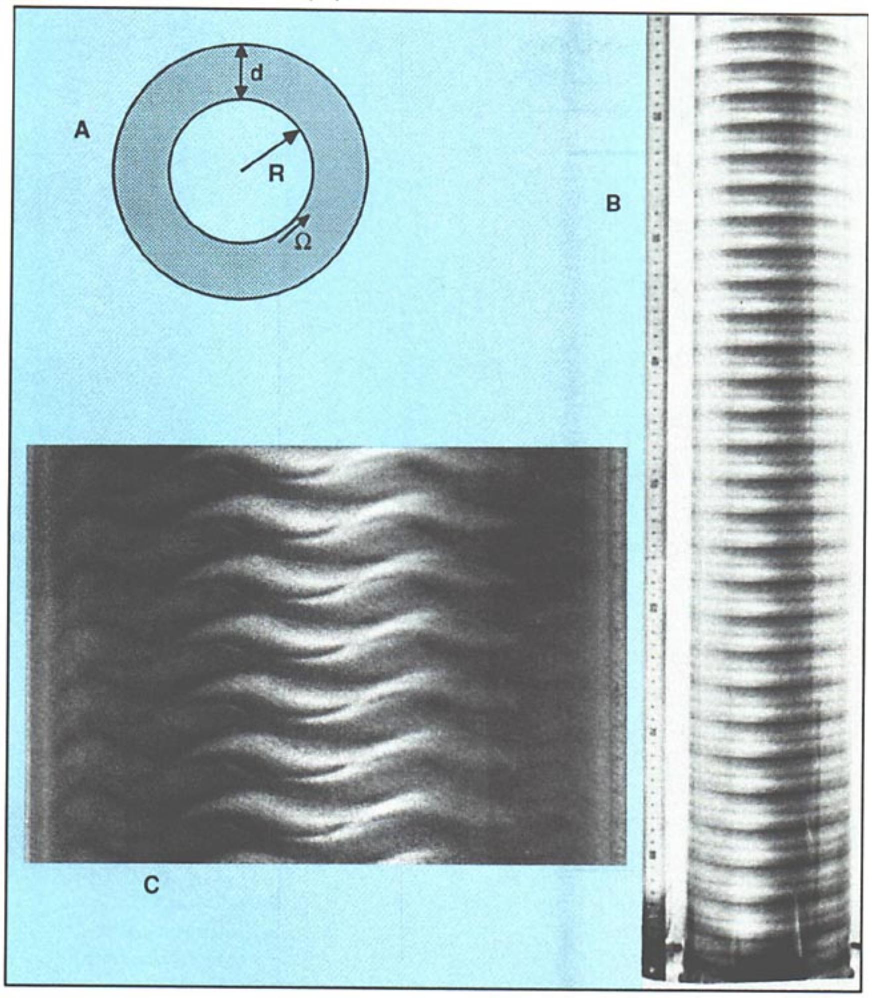
The first study of the instability of rotating flows was done by Rayleigh (*8*) more than a century ago. Rayleigh's criterion for stability of a circulating inviscid fluid simply states that

$$\Phi\,(r) \equiv (1/r^3)\,d\,(r^2\,\Omega)^2/dr > 0 \qquad (16)$$

everywhere in the field of flow. Here $\Phi\,(r)$ is the Rayleigh discriminant and the angular velocity of the fluid, $\Omega\,(r)$ is an arbitrary function of the position r from the axis of rotation. This general criterion is limited to axisymmetric disturbances. The transition in circulating flows is mainly based on centrifugal instability. To illustrate, we will review Rayleigh's physical discussion of centrifugal instability for an inviscid fluid.

The radial momentum balance for an axisymmetric flow states

6. Flow between concentric rotating cylinders: (A) schematic of cross section, (B) steady Taylor vortices, (C) time-periodic state (*15*)



$$\frac{D}{Dt}\,(ru_\theta) = 0 \qquad (17)$$

where D/Dt is a material (substantial) derivative and u_θ ($= r\Omega$) is the angular component of velocity. This equation simply states that the angular momentum of a fluid element, $r\,u_\theta$, is constant. The centrifugal force density in the radial direction is $\rho(H^2/r^3)$, and the kinetic energy is $(1/2)\rho(H^2/r^3)$.

Now consider two fluid elements rotating around the axis of rotation with an angular velocity at arbitrary positions $r = r_1$ and $r = r_2$, respectively. Let us assume that the fluid elements have an equal volume, δV. Defining this state as Stage b, then the total kinetic energy of the elements due to azimuthal motion is given by

$$KE_b = \frac{1}{2}\,\rho\left(\frac{H_1^2}{r_1^2} + \frac{H_2^2}{r_2^2}\right)\delta V \qquad (18)$$

If we interchange the fluid elements such that Element 1 is rotating with radius r_2 and Element 2 is rotating with radius r_1, defining this state as State a, then the total kinetic energy of the system will be equal to

$$KE_a = \frac{1}{2}\,\rho\left(\frac{H_1^2}{r_2^2} + \frac{H_2^2}{r_1^2}\right)\delta V \qquad (19)$$

The difference in the kinetic energy from States b to a is given by

$$KE_a - KE_b = \frac{1}{2}\,\rho\,(H_2^2 - H_1^2)\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)\delta V \qquad (20)$$

If $r_2 > r_1$ and $H_2 < H_1$ (i.e., if $dH/dr < 0$, a violation of Rayleigh's criterion for stability), the net change in kinetic energy due to the interchange of fluid elements is negative (State a requires less kinetic energy). Therefore, the rotating flow is unstable. This result is limited to axisymmetric disturbances. The stability properties with respect to more general disturbances are treated by Synge (*14*).

In general, viscosity has a stabilizing effect in a rotating flow. Therefore, if a basic flow is stable according to Rayleigh's criterion, then it must remain stable as viscosity is introduced into the system. On the other hand, if a system does not satisfy Rayleigh's criterion for stability, then adding viscosity may or may not have a stabilizing effect.

The Taylor-Couette problem

Perhaps the most popular example of a system that exhibits centrifugal instability is the Taylor–Couette cell in which fluid is contained between two concentric rotating cylinders (**Fig. 6**). This system was analyzed by Taylor (*16*), who obtained the curve of marginal stability in the plane of the angular velocity of the inner and outer cylinders. The two-dimensional basic flow destabilizes at a critical Reynolds number and gives rise to doughnut-shaped vortices, sometimes referred to as "Taylor vortices" (Fig. 6B).

The Reynolds number for this problem can be defined as

$$R_T = R(d/R_1)^{1/2} \qquad (21)$$

where

R = $R_1\,\Omega\,d/\nu$

d = gap between the two cylinders

R_1 = radius of the inner cylinder

Ω = angular velocity of the inner cylinder.

The critical Reynolds number for the onset of Taylor vortices in an ideal system is about 41.3. As the Reynolds number is increased, a second transition replaces the steady-state flow with a time-periodic state (Fig. 6C). For an infinitely long cylinder, these vortices are perfectly uniform and periodic in the axial direction. A cylinder with finite length, however, introduces some imperfections as a consequence of the end boundaries.

Burkhalter and Koschmieder (*17*) showed that in a long circular Couette cell, resting end walls fixed with the resting outer cylinder produce a sink at the boundary; that is, the fluid moves inward near the boundaries. On the other hand, when end walls rotate with the inner cylinder, then a source generates at the end boundaries. Also, multiple stable states are shown to exist in systems of finite length.

A closely related case is the Taylor-Dean problem, in which, in addition to the rotation of the cylinder, an azimuthal pressure gradient is also present (*18*, *19*). Similar to the Taylor problem, the instability is made evident by the appearance of toroidal vortices. Raney and Chang (*20*) showed that the system is also unstable to oscillatory disturbances.

The Dean problem

A pressure-driven flow in a curved channel (**Fig. 7**) can also exhibit centrifugal instability. The first study of this problem is by Dean (*21*), who considered the flow in a narrow channel formed by two coaxial cylinders. For a constant pressure gradient acting in the azimuthal direction and with the *narrow-gap approximation*, the velocity distribution is given by the parabolic form of the plane Poiseuille flow:

$$v(r) = \left\{ 1.5 - \frac{6}{d^2}\left[r - \frac{1}{2}(R_1 - R_2)\right]^2 \right\} V_o \quad (22)$$

where

$d \;=\; R_2 - R_1$

$V_0 \;=$ mean velocity across the channel.

The Reynolds number for this problem (sometimes referred to as the "Dean number") is given by

$$R_D = R(d/R_1)^{1/2} \quad (23)$$

where

$R \;=\; V_0\,(d/\nu)$.

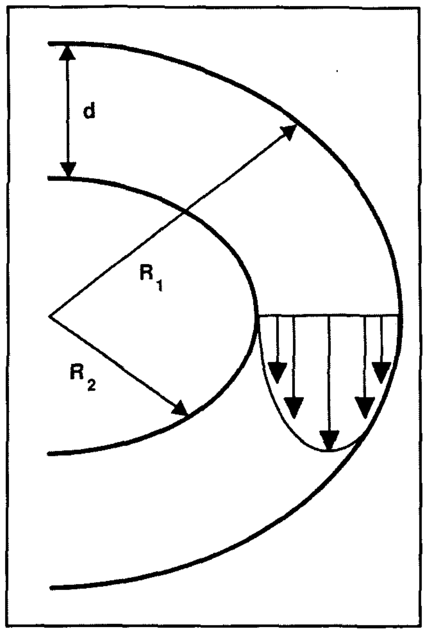
The flow represented by the velocity profile $v(r)$ can destabilize by two entirely different mechanisms. In addition to the centrifugal instability, which necessarily includes three-dimensional disturbances, this flow, an approximately parallel shear layer, can also destabilize to two-dimensional disturbances. The question is, when would each instability mechanism be the dominant one? To answer this question, let us consider the critical parameters for the onset of each instability.

The critical value of the Dean number, R_D, signaling the onset of three-dimensional disturbances, is about 35.9. On the other hand, the critical Reynolds number, R_C, for the instability of plane Poiseuille flow to two-dimensional disturbances is about 5772 × (4/3). Therefore, when

$$R_C(d/R_1)^{1/2} < 35.9$$

(i.e., when $d/R_1 < 2.17 \times 10^{-5}$), the first supercritical state of the flow remains two-dimensional; otherwise three-dimensional vortices appear as a result of centrifugal instability.

7. Pressure driven flow in a curved channel



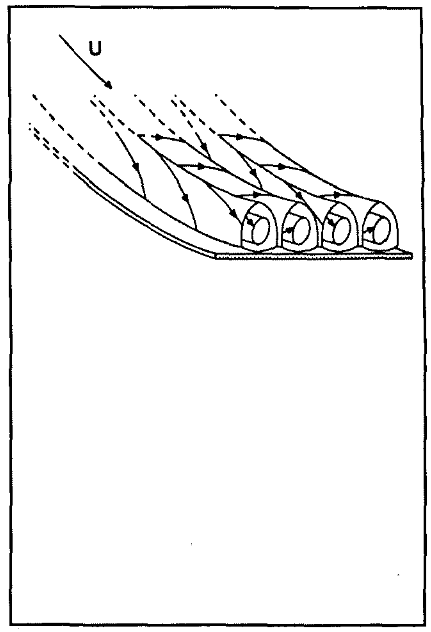
Therefore, for a pressure-driven fluid layer in a curved channel to behave as a parallel shear flow, the ratio for the gap to the radius of curvature has to be less than about 2×10^{-5}. This rather general conclusion will be useful as a guide in various coating flows that may exhibit potential for both kinds of instability mechanisms.

Görtler instability in boundary layer flows

A third classic problem that exhibits centrifugal instability is the formation of a thin viscous boundary layer over a concave wall. By definition, a boundary layer over a solid surface is a thin layer across which the fluid velocity increases from zero (no-slip) at the wall to the free-stream velocity. Viscous effects are then important only inside the boundary layer. This condition appears only when the Reynolds number based on the free stream velocity is sufficiently large.

Boundary layer flow over a concave solid surface is prone to centrifugal instability, which results in the formation of counter-rotating toroidal vortices with axes of rotation parallel to the direction of the base flow (**Fig. 8**). These vortices are referred to as "Görtler vortices." In Görtler's analysis (*22*), it was assumed that the boundary layer thickness is much smaller than the radius of curvature of the wall and that the base flow is nearly parallel to the wall; therefore, centrifugal effects are important only in the boundary layer region.

8. Görtler vortices in a boundary layer along a concave wall



It is then easy to show that the Rayleigh discriminant in this problem is proportional to -V (dV/dy'), where $y' = y/\delta$, δ is the boundary layer thickness, and the fluid velocity V increases inside the layer from $y = 0$ at the solid surface to $y = \delta$ at the free stream. The Rayleigh discriminant is therefore always negative, which indicates that the flow can become unstable in the boundary layer.

Looking ahead

In Part II, we shall analyze some examples of flow instability in coating systems, along with more classical flow instability problems. Some generalizations of the behavior of dynamical systems will be discussed in Part III.□

Literature cited

1. Landau, L. D. and Lifshitz, E. M., Fluid Mechanics, Vol. 6, Pergamon Press, London, 1959, p. 102.

2. Reynolds, O., Phil. Trans. Roy. Soc. 174: 935(1883).
3. Aidun, C. K. and Triantafillopoulos, N. G., Global stability properties of flow structures in the pond of a short-dwell blade coater, Intl. Symp. Mech. of Thin-Film Coating, AIChE, New York, 1990.
4. Kistler, S. F. and Scriven, L. E., Coating flows *in* Computational Analysis of Polymers (J. Pearson and S. Richards, Eds.), Ch. 8, Barking, Essen. Engl.; Appl. Sci. 343: (1983).
5. Rayleigh, Lord, Proc. London Math. Soc. 10: 4(1879).
6. Rutland, D. F. and Jameson, G. J., J. Fluid Mech. 46: 267(1971).
7. Lin, C. C., The Theory of Hydrodynamic Stability, Cambridge University Press, 1955.
8. Rayleigh, Lord, Proc. London Math. Soc. 11: 57(1880).
9. Fjortoft, R., Geofys. Publ., Oslo 17(6): 1(1950).
10. Squire, L. C., Proc. Roy. Soc. A 142: 621(1933).
11. Orr, W. M. F., Proc. Roy. Soc. Irish Acad. A 27: 9(1907).
12. Sommerfeld, A., Ein Beitrag zur hydrodynamischen Erklaerung der turbulenten Fluessigkeitsbewegungen, Proc. 4th International Cong. Mathematicians, Rome, Vol. III, 1908, p. 116.
13. Orszag, S.A., J. Fluid Mech. 50(4): 689(1871).
14. Synge, J. L., Semi-centenn. Publ. Amer. Math. Soc. 2: 227(1938).
15. Koschmieder, E. L., 1979, J. Fluid Mech. 93: 515(1979).
16. Taylor, G. I., Phil. Trans. Roy. Soc. A223: 289(1923).
17. Burkhalter, J. E. and Koschmieder, E. L., J. Fluid Mech. 58: 547(1973).
18. Brewster, D. B. and Nissan, A. H., Chem. Eng. Sci. 7: 215(1958).
19. Brewster, D. B., Grosberg, P., and Nissan, A. H., Proc. Roy. Soc. A251: 76(1959).
20. Raney, D. C. and Chang, T. S., Z. Agnew Math. Phys. 22: 680(1971).
21. Dean, W. R., Proc. Roy. Soc. A 121: 402(1928).
22. Görtler, H., On the three-dimensional instability of laminar boundary layers on concave walls, Nachr. Ges. Wiss. Göttingen 2(1): (1940).

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