Roll-structure calculations at constant wound-in tension

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ABSTRACT *Roll-structure theory is used to calculate internal pressures and tensions in a roll or reel wound at constant tension. The calculation is able to theoretically predict the interlayer pressure at the core, which is of interest in the case of gearing effects. The calculation also predicts the relationship between the roll-density monitor and wound-in tension. The equations used are consistent with those of Altmann and Hakiel, and they incorporate Pfeiffer's nonlinear stress–strain relationship for compression perpendicular to the sheet. However, the computational problem was simplified by assuming constant wound-in tension and by ignoring localized core effects. The resulting equations are easy to solve on a personal computer and are intended as a compromise solution to gain a qualitative understanding of roll-structure calculations with a minimum of computational effort. Results for both paper and plastic film are compared with results obtained from models developed by Pfeiffer and Hakiel. Theoretical predictions of the corresponding roll density also are presented.*

KEYWORDS
Analysis
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Winding
Winding tightness
Wound rolls

Calculation of pressure and tension inside a wound roll is of some practical value in determining whether the roll is susceptible to defects, such as gearing at the core (*1*). It is also of interest in relating test results from different measurement devices, such as the WIT/WOT (wound-in tension/wound-off tension) machine (*2*) and the roll-density monitor (*3*).

Unfortunately, from a theoretical point of view, the problem is not easy to solve. Not only are the differential equations nonlinear (*4*), they are also inherently two dimensional in nature (*5*). This is because some features of roll structure, such as core effects (*6*), show up at a fixed radius, while other features, such as the tension band at the surface, show up at a variable radius that grows with the roll as it is wound.

The present work is an attempt to gain some qualitative understanding of the problem with a minimum of computational effort. We will deal rigorously with the nonlinear compressibility of the paper or film while ignoring the core effects, which are often so localized that they are not only difficult to observe experimentally but also difficult to simulate theoretically. The result is a set of equations that are very easy to solve and which give reasonable predictions of properties such as the core pressure and the roll-density monitor. The major practical shortcoming of the present calculation is that it was necessary to assume a constant wound-in tension, which makes the simulation more appropriate to a paper machine jumbo reel than to a winder roll.

Theory

We wish to develop a theory that will be compatible with that of Hakiel (*5*) but will be easier to solve. The stresses in a roll are the interlayer pressure, P (psi, positive when the web is compressed), and the tension, T (psi, positive when the web is stretched). These are functions of the radius, r, where they are measured, as well as the full roll radius, R. We wish to investigate the consequences of assuming that they are functions only of the dimensionless ratio $u = r/R$. That is, the shape of the stress profile of the roll remains the same as the roll is built. The stress equilibrium equation (*7*, Eq. 1) becomes

$$P + T + uP' = 0 \tag{1}$$

where the prime sign (′) denotes d/du.

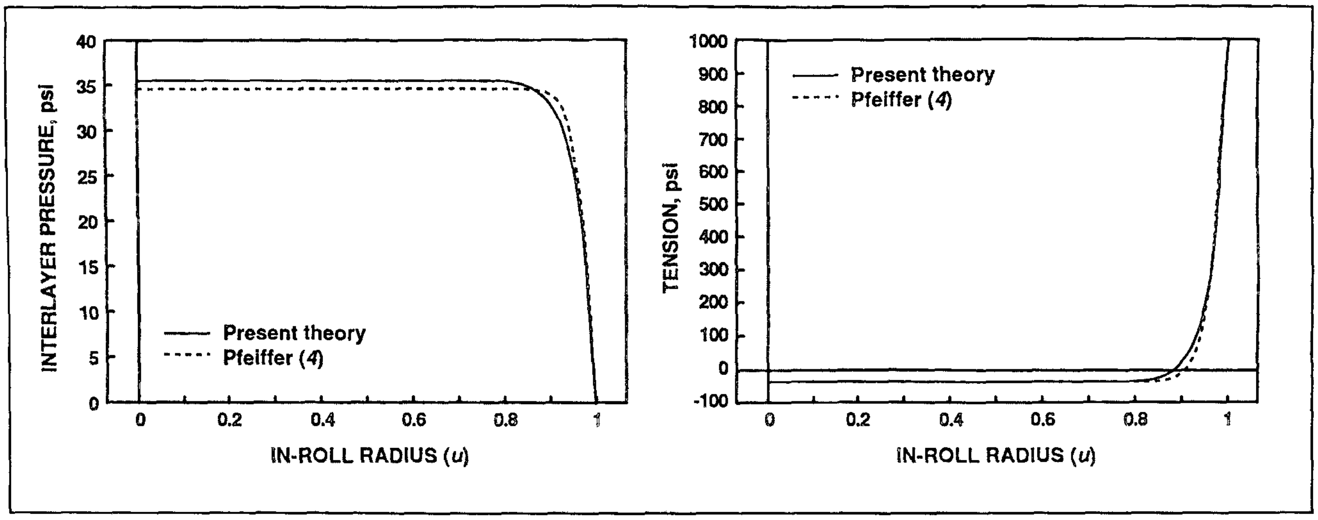
The strain compatibility equation (*7*, Eq. 7) becomes

$$[(E_t/E_r)\,(dP/du)] + \{(d/du)[T + u(dT/du)]\} = 0 \tag{2}$$

where

E_t = tangential modulus

1. Predicted stresses for a roll of paper wound at a tension of 3 pli



E_r = radial modulus.

We will use Pfeiffer's (8) radial modulus

$$E_r = K_2(P + K_1) \quad (3)$$

where K_1 and K_2 are material-dependent compressibility constants, with K_2 sometimes called the "springiness factor." Tangential modulus E_t is assumed constant.

Note that Eq. 2 is an ordinary differential equation and not a partial differential equation, as in Penner (7, Eq. 7). This is the major simplification achieved by assuming stresses are functions only of u. The drawbacks will become apparent later, in connection with boundary conditions. Equation 2 can be integrated to yield

$$(E_t/K_2)\ln[1+(P/K_1)] + T + uT' = C \quad (4)$$

where C is an unknown constant.

Equation 4 can be combined with Eq. 1 to yield an equation for P.

$$(u^2P'') + (3uP'') + P + \{(E_t/K_2)\ln[1 + (P/K_1)]\} = -C \quad (5)$$

Note the similarity with equations presented by Hakiel (5, Eq. 5) and by Willett (9, Eq. 9) if Poisson's ratio is zero, as we will assume. The main difference is the presence of the constant C in Eq. 5, which plays a critical role in the solution of the one-dimensional problem.

The boundary conditions at the roll surface ($u = 1$) are $P = 0$ and $T = T_w$ (wound-in tension). This last condition can be satisfied only if T_w is constant. With these two conditions at the surface, the only remaining unknown is C. C must be determined by a boundary condition at the core. However, the core boundary condition (7, Eq. 8) generally cannot be satisfied because it is imposed at a constant r and not at constant u. We can only satisfy it in a trivial way by assuming there is no physical core and by assuming $P' = T' = 0$ at $u = 0$. We will continue to refer to the point $u = 0$ as the "core" for lack of a better word. With these conditions at the origin, we can relate C to the core pressure P_0 = P (at $u = 0$) by Eq. 6

$$C = \{(E_t/K_2)\ln[1 + (P_0/K_1)]\} - P_0 \quad (6)$$

which follows from Eq. 5. Equation 6 is helpful in interpreting the results, but it is not essential to the solution method. In practice, one can guess a value for C and solve Eq. 5 inward from the surface to see the behavior at the origin. The solutions turn out to be extremely unstable at the origin, with pressure spikes that can be either positive or negative infinity. There is only one value of C that is able to eliminate the spikes, and it happens to satisfy Eq. 6, as expected. The numerical technique for doing this guesswork automatically is described in the appendix to this study.

The boundary conditions described so far are adequate for the simulations presented in the following pages. However, in general, the core boundary condition may actually be too restrictive. The real condition we wish to satisfy is energy conservation, which is expressed by

$$2\int_0^1 (D_t + D_r)\, u\, du = T_w^2/2E_t \quad (7)$$

where D_t and D_r are tangential and radial energy densities generated by P and T (7, Eqs. 9 and 10). Using the procedure outlined by Penner (7), it can be shown that energy is automatically conserved for the whole roll if the following condition is met at the origin:

$$u^2(D_t + D_r) \longrightarrow 0$$

This condition is less restrictive than the condition that $P' = T' = 0$ at the origin. However, the distinction is not important in the results presented here.

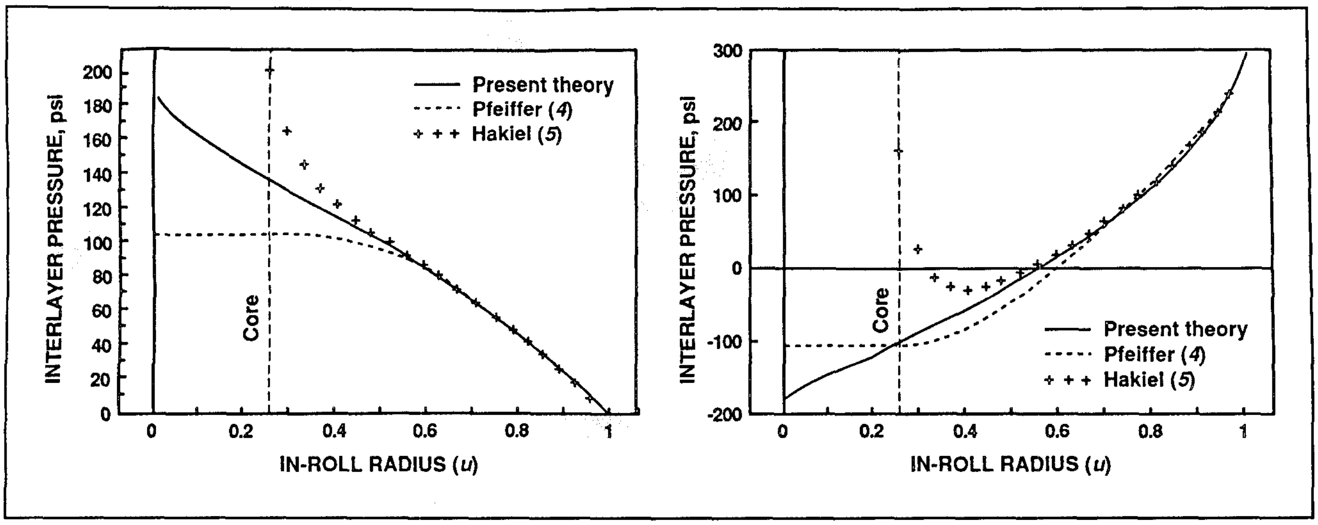
Stress profiles

Figure 1 shows pressure and tension calculated inside a roll of paper, as a function of the dimensionless radius u. The parameters are

K_1	=	15 psi
K_2	=	11
E_t	=	340,000 psi
T_w	=	1000 psi (3 pli)
Caliper	=	0.003 in.

Figure 1 shows that the roll develops a core pressure of about 35 psi, and it is clear that the pressure and tension are constant near the origin, as was assumed in the previous derivation. Also shown is the result of Pfeiffer's calculation (4), which agrees quite well with the present theory. No attempt was made to generate core

2. Predicted stresses for a roll of film wound at a tension of 0.9 pli. Core applies only to Hakiel data.



effects using Pfeiffer's theory, since these effects would be highly localized (*6*), probably within half an inch of the core for a 40-in.-diam. roll. The reason is that the paper is highly anisotropic (high E_t/E_r).

A much more difficult test of the theory is shown in **Fig.** 2, which is a simulation of a polyester film roll, with the parameters

K_1	=	1 psi
K_2	=	1060
E_t	=	600,000 psi
T_w	=	300 psi (0.9 pli)
Caliper	=	0.003 in.

In this case, the radial compressibility is much more nonlinear, and the anisotropy is not nearly as high as for paper, both of which complicate the calculation. The simulation essentially duplicates results obtained by Hakiel (*5*, Fig. 3), with minor changes to accommodate the compressibility model in Eq. 3. The present theory and Pfeiffer's theory are shown, both with no core effects. Hakiel's numerical simulation contains a core, located at $u = 0.25$, with a modulus of 890,000 psi. It is clear that the core effect is quite important in this case, but it does not extend to the surface of the roll.

Roll-density monitor

The main motive for developing the present, approximate theory was to study the roll-density monitor (RDM), which measures a surface property of the roll. We wish to relate measurement of RDM to measurement of the wound-in tension from a WIT/WOT machine to see whether the two measurement techniques are somehow equivalent.

Penner (*7*) has shown that, for a linear radial compressibility, the roll density is given by

$$\mathrm{RDM} = 1 + [T_w/(E_t\, K_1K_2)^{1/2}] \qquad (8)$$

while the core pressure is given by Altmann (*10*)

$$P_0\,(\text{linear}) = T_w/(g^2 - 1) \qquad (9)$$

where

$g^2 \;=\; E_t/(K_1K_2)$

In both expressions, we have ignored core effects and have assumed $K_1 >> P$ at all times. The core pressure has been included in this discussion because it is of equal practical significance to roll density. We now wish to study these two properties of the roll for the nonlinear case.

In general, roll density can be defined (*7*) to be

$$\mathrm{RDM} = 1/\{1 - [T'(1)/E_t]\} \qquad (10)$$

where

$T'(1)$ = slope of the tension band at the surface

There are no explicit boundary conditions that directly determine T', but it can be related to C by the expression

$$C = T_w + T'(1) \qquad (11)$$

which follows from Eq. 4 with $P(1) = 0$. Therefore, roll density is given by

$$\mathrm{RDM} = 1/\{1 - [(C - T_w)/E_t]\} \qquad (12)$$

This expression incorporates the effects of nonlinear compressibility and is probably quite reasonable even when core effects are present.

As an aside, note that Eq. 6 contains an expression for P_0 in terms of C, assuming there is no core. We therefore have a direct link between a "core" property and a surface property of the roll, which illustrates the boundary-value nature of the problem.

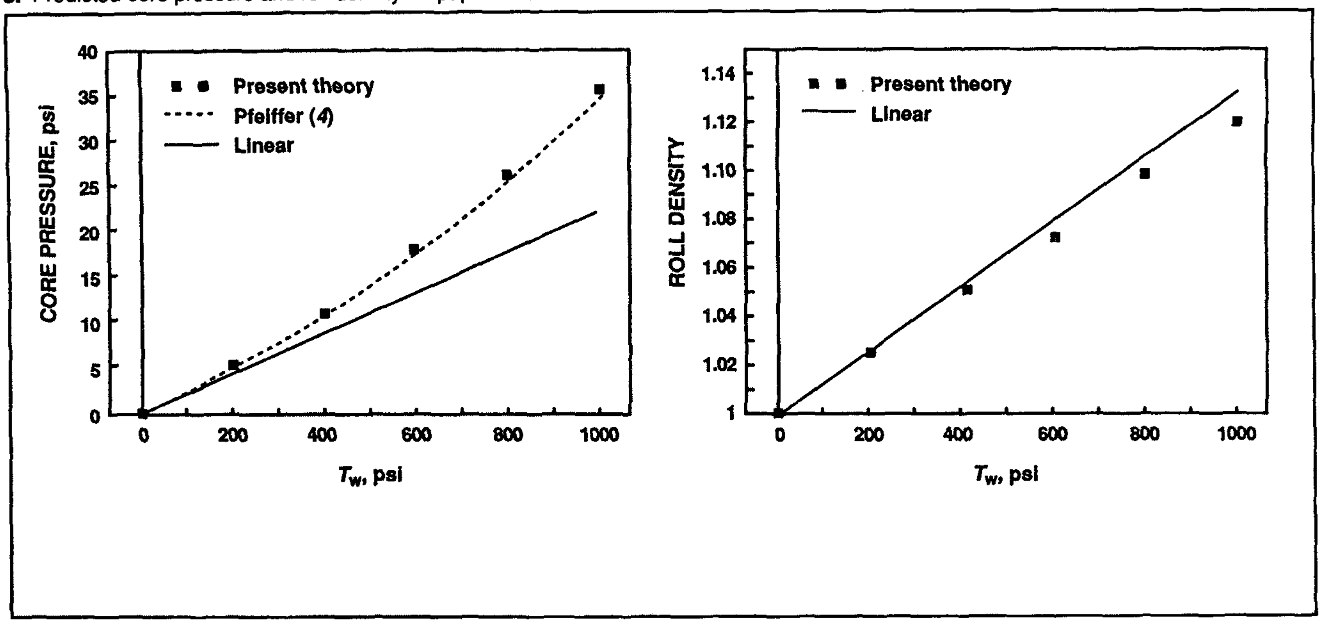
Figure 3 shows calculated core pressure, P_0, and roll density for a series of paper rolls, each wound at a different tension. The paper properties are as in Fig. 1, but with different values of T_w. The predictions of the linear theory (Eqs. 8 and 9) are shown, as is Pfeiffer's predicted core pressure (assuming no core effects). This is obtained by solving the equation

$$D_r + (D_rD_t)^{1/2} + D_t = T_w^2/2E_t \qquad (13)$$

with the constraint that $P + T = 0$ at the origin (Eq. 1). Solving for P gives a rapid estimate of core pressure that agrees very well with the present theory, in the case of paper rolls.

Figure 3 shows that the nonlinear compressibility leads to a core pressure higher than expected and a roll density less than expected. Both of these trends can be rationalized from the linear equations (Eqs. 8 and 9) if one allows the effective value of E_r (or K_1K_2) to increase due to nonlinearity.

3. Predicted core pressure and roll density for paper rolls wound at various tensions



4. Predicted core pressure and roll density for film rolls wound at various tensions

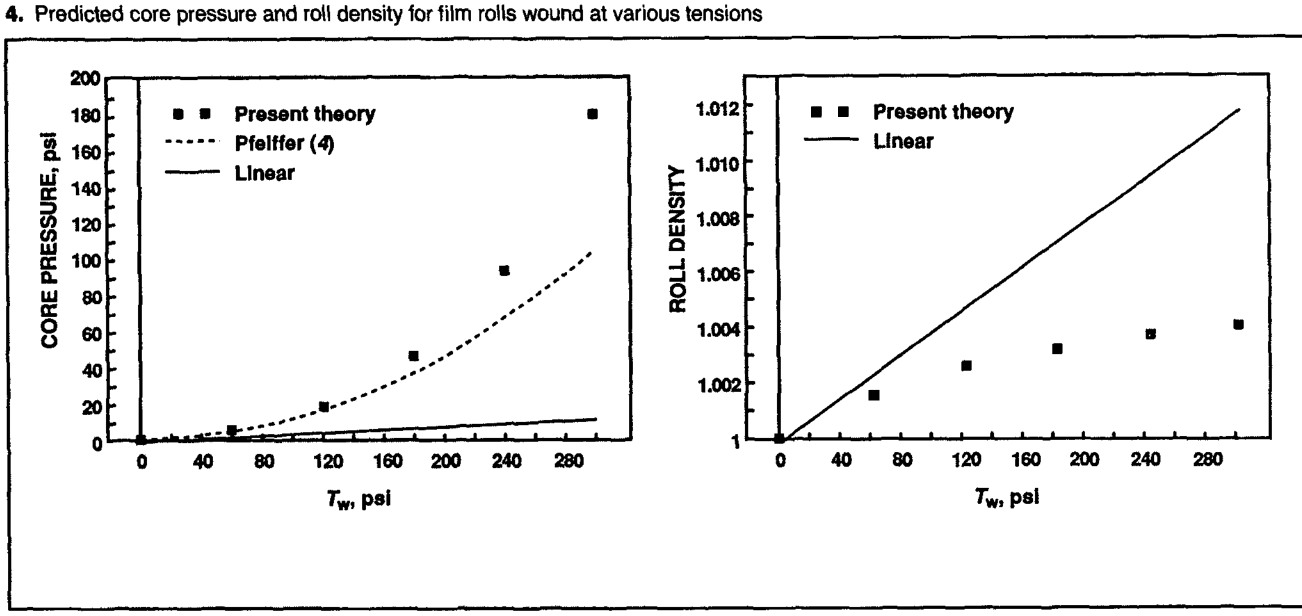


Figure 4 shows the same quantities calculated for a series of film rolls, wound at different tensions. The film properties are those of Fig. 2, but again with different values of T_w. In this case, the predictions of the linear theory are of no use. It is interesting to note, however, that the linear theory does a better job of estimating roll density than it does in the estimation of core pressure.

It appears that the qualitative behavior of roll density will be very similar to that of wound-in tension, even when the compressibility behavior is highly nonlinear. This is a result that would have been difficult to predict analytically, and it gives some encouragement for the use of RDM monitors to evaluate wound-in tension on-line.

Appendix

The numerical method used here is probably best described as a "shooting" method (*11*, p. 359), as opposed to a band matrix technique (*5*, *9*). Assume that we have an estimate of P_0 that satisfies Eq. 6 and thus provides a value for C. Solve Eq. 5 as an initial-value problem by dividing the interval $u = (0,1)$ into N steps of length h. Replace P' and P'' by standard finite-difference expressions; let $P_N = 0$ at the surface; approximate P_{N-1} by a quadratic Taylor expansion at the surface; and iterate inward to the core. In our case, $N = 600$, which was convenient for graphing purposes.

As the solution approaches the origin, it will normally diverge very rapidly in either direction. Too high an estimate of P_0 will generate a negative pressure spike, and vice versa. It is impossible to remove the spike completely, but it is possible to minimize its strength to the point where the remainder can be attributed to round-off error in the computer. We therefore need an algorithm to estimate the size of the pressure spike.

Assume that the pressure spike originates in the vicinity of a pressure P_0 that satisfies Eq. 6. Approximate Eq. 5 by a linear expansion around this pressure, namely

$$(u^2P'') + (3uP') + [(1 - g^2)(P - P_0)] = 0 \quad \text{(A1)}$$

where

$$g^2 = E_t/K_2(K_1 + P_0)$$

In the vicinity of P_0, the solutions are therefore given by

$$P = P_0 + Au^{-1-g} + Bu^{-1+g} \quad \text{(A2)}$$

where A and B are unknown. This is essentially Altmann's solution, except that in our case, it is valid only locally and is of no use whatsoever in predicting the overall stress distribution. It is of use only for estimating the ratio A/B, which in turn will allow us to refine our estimate of P_0.

It is clear that we wish to estimate P_0 in such a way that $A/B = 0$ in Eq. A2, because the term involving A will invariably diverge at the origin. Two cases arise:

If P_0 is too small, then the calculated pressure, P, obtained from solving Eq. 5 inward, will at some point, u, cross the core pressure, P_0, and continue upward into a positive spike. In this case, Eq. A2 yields the estimate

$$A/B = -u^{2g} \quad \text{(A3)}$$

where u is the position of the crossover.

If P_0 is too large, then the calculated pressure, P, will never reach P_0 as we move inward but will instead go through a maximum value and turn downward to a negative spike. Setting the slope of P in Eq. A2 equal to zero yields the estimate

$$A/B = [(g - 1)/(g + 1)]\, u^{2g} \quad \text{(A4)}$$

where u is now the position of the maximum. (Note that this entire procedure is reasonable only if $g > 1$, which is certainly true in the present simulations but may not always be true at very high wound-in tension or core pressure. If it is not true, then the role of energy conservation, Eq. 7, must be reevaluated.)

Numerical simulations have shown that the ratio A/B is a reasonably smooth function of P_0, despite the fact that the function has two branches that must be evaluated separately. This ratio therefore can be used to estimate P_0 such that $A/B = 0$. The numerical procedure is as follows: Estimate P_0 and calculate C from Eq. 6. Solve Eq. 5 inward and use Eqs. A3 or A4 to calculate A/B at two different values of P_0. Use the secant method (*11*, p. 227) to refine the estimate of P_0 until convergence has been obtained. The procedure is initialized by using Pfeiffer's energy balance (Eq. 13) for the first estimate of P_0. It normally converges in about 10 iterations, taking a total of about 30 s. The calculations were done on a personal computer with a BASIC compiler and an arithmetic coprocessor. Both are desirable in this case, for the sake of speed and double-precision arithmetic.☐

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