Paper deformation in a calendering nip

José J. A. Rodal

ABSTRACT: Numerical calculations (finite-element analysis) of plastic (permanent) deformation of a paper web during soft-nip calendering are presented. Paper is modeled as a porous, cellular, solid material. Stress and deformation results are discussed in relation to theories previously presented in the literature. Emphasis is placed on the relative role of hydrostatic (mean) and shear stresses in producing plastic deformation of the paper web.

KEYWORDS: Bibliographies, calendering, computation, finite element analysis, paper webs, plastic deformation, soft nip calenders, stresses.

An excellent review of the technology of paper calendering is available in the literature *(1)*. Peel wrote this thorough work, which includes 135 references covering work up to 1989. Others who have contributed greatly to this technology include (in alphabetical order) Baumgarten, Crotogino, Kerekes, and Pfeiffer.

The work presented here discusses the deformation of paper in a calendering nip, with emphasis on the relative roles of hydrostatic and shear stresses, and the permanent change in volume experienced by the paper sheet. Several models of paper deformation are examined.

One of the greatest scientists of this century, John von Neumann, provided the following definition of a scientific model:

> "The Sciences do not try to explain, they hardly even try to interpret, they mainly make models. By a model is meant a mathematical construct which, with the addition of some verbal interpretations, describes observed phenomena. The justification of such a construct is solely and precisely that it is expected to work."

Deformation in calendering nip

Hydrostatic stresses

One may question whether there is a need to consider hydrostatic pressure in the calendering process *(2)*. We may start by considering Hertz's *(3)* solution for the contact of two cylinders in plane-strain. It is known from Hertz's solution that (a) the point located under the peak contact pressure experiences in-plane principal stresses that are equal to each other and (b) the third principal stress is equal to the sum of the in-plane principal stresses times the Poisson's ratio *(4–6)*. Therefore, the in-plane principal shear stress (proportional to the difference between the in-plane principal stresses) is zero directly under the peak pressure for plane-strain Hertzian contact.

The use of the von Mises yield criterion and its associated flow rule imply a plastic Poisson's ratio of 0.5. Hence, for plastic plane-strain, the region directly under the peak pressure is purely hydrostatic *(3)*. Of course, the stresses away from this area cease to be purely hydrostatic, and there is shear as well, but the hydrostatic component continues to be very important and is certainly not negligible. I have shown color contour plots of the shear stress distribution on a cover in a calender nip based on finite-element analysis *(7, 8)*.

We also should consider that there is a region of hydrostatic stress that is large with respect to the principal shear stresses in most contact problems. If one uses a plastic-yield condition that is independent of hydrostatic stresses (e.g., the von Mises criterion) to analyze a material in plane plastic-strain contact, the results will show a region that is purely elastic, since the material cannot plastically flow in that region. If one uses the popular assumption that plastic strains are much larger than elastic strains *(9)*, such that one can model the elastic regions as being rigid relative to the plastic regions, then there is a "rigid zone" in these problems. There are a number of mathematical solutions of these problems. These regions are named the "hydrostatic core," "rigid core," or "elastic core." Johnson and Mellor show a number of beautiful graphs of these "hydrostatic" or "rigid" cores in

Rodal is president of Advanced Materials Corp., 10 Industrial Hwy., Philadelphia, PA 19113-2004.

their book *(9)* on plasticity, which is a classical reference on plasticity of metals.

Plastic-yield criteria

Pfeiffer *(2, 10)* advocates the use of the von Mises yield criterion, and furthermore claims *(10)* that "it is the only way to indicate how close a part is to failure using a color scheme." This claim is incorrect, since there are a number of plastic-yield criteria *(9, 11–15)* available to determine the elastic limit of a material. All of these yield criteria can be plotted with a color scheme to represent the value of the yield function if one chooses to do so. Even in the case of an anisotropic yield function (i.e., one with a number of material constants to model different material properties in different material directions), there is no problem in displaying it as a color contour plot, since there is only one value of the yield function at a given material point!

In the case of the von Mises yield criterion, the assumptions are that the material is completely isotropic (i.e., the material properties are independent of orientation) and that hydrostatic stresses cannot affect the elastic limit *(15)*. Therefore, only the principal shear stresses can produce plastic yielding of a material under the von Mises criterion. Von Mises *(16)* initially proposed his criterion as a simpler alternative to the Tresca *(17)* and the Mohr–Coulomb *(18, 19)* criteria, which at that time were believed to be the ones that most accurately represented plasticity of materials. Quoting from the original paper by von Mises *(16)* (my translation from the German language for the text immediately after Eq. 25):

"In any case, equation 25 [the von Mises yield criterion], allows a much simpler analytical treatment, and the difference with respect to equation 24 [which is now known as the Tresca yield criterion] is not much greater than the range of certainty of experiments performed up to the present time." And earlier in the paper (point b on p. 586), "If the elasticity limit is reached, then the solid behaves essentially like a viscous and incompressible liquid."

Notice that von Mises envisions the body being plastically deformed as a fluid, which is therefore incompressible. However good a model this is for metals, it is not a good model for plastic deformation of a cellular material like paper, which has a large percentage of its volume occupied by air and therefore is very compressible.

The distortion-energy interpretation of the von Mises yield criterion is sometimes advocated *(2, 10)*. This interpretation predicts plastic yielding when the elastic distortion energy (the part of the strain energy that is independent of hydrostatic stresses) reaches a critical value. This interpretation was not promoted by von Mises, but it was advanced by Hencky *(20)* more than ten years after von Mises' paper. As Hill *(15)* has pointed out, "Hencky's (distortion energy) interpretation does not seem to have a general physical significance" because one cannot uniquely define a distortion energy for generally anisotropic bodies. This is because, for some kinds of anisotropic materials, a hydrostatic pressure will produce elastic distortions in addition to a change in volume. Furthermore, the distortion energy is an elastic concept (being dependent on the value of the elastic shear modulus), while plasticity is produced in metals by the movement of dislocations, which are irrecoverable and hence inelastic in nature.

It seems preferable to use a physical interpretation that is independent of elastic constants. One such physical explanation—the octahedral shear stress interpretation originally advanced by Eichinger *(21)*—is that the von Mises yield criterion predicts that plastic yield will occur when the shear stress, acting on the plane forming equal angles with each of the principal stress axes, reaches a critical value.

It is true that for most metals (with the exception of annealed mild steel, cast iron, and other metals that should be familiar to engineers in the paper industry), the von Mises yield criterion is able to fit the yield limit as well as or better than more complicated criteria. This reason, combined with its relative simplicity, makes it the most popular yield criterion for plastic deformation analysis of engineering metals. Such metals have a very orderly crystalline structure, and plastic flow takes place by moving dislocations along slip planes. Slip is their primary plastic deformation mechanism, and for metals with a cubic crystalline structure, it can operate equally well forward and backward. (This explains the symmetry, with respect to plastic yield in tension and compression, displayed by these materials). Therefore, plastic flow for such materials can be very well modeled as distortion without change in volume. In this sense, plastic flow of metals has some similarity to the flow of fluids, as originally envisioned by von Mises *(16)*, from the point of view that fluids flow by distortion and are incompressible.

Another ingredient necessary for a plasticity model, besides the yield criterion, is a stress–strain relation for the plastic range: an expression enabling us to obtain the plastic strain rates from the stresses. This expression can be obtained from the yield criterion itself by postulating that the plastic strain rate vector is in the direction of the outward normal to the yield surface (in stress space). According to this postulate, the components of the plastic strain rate are proportional to the gradient of the yield criterion with respect to the stress components. The relevance of this is that a choice of yield criterion has important consequences, not only in determining the limit of elastic deformation, but also in determining the direction of the plastic strain rates.

Permanent volume deformation

The von Mises criterion assumes isotropy of the most general kind, not only as to the orientation of stresses with respect to the body, but also as to the sign of the stress (whether tensile or compressive). Furthermore these assumptions, coupled with the hydrostatic-pressure-independence assumption, imply that there are no volume changes in the plastic regime!

When Pfeiffer *(2, 10)* states that paper in a calender nip can be modeled by von Mises' yield criterion, he is implicitly stating that paper can be modeled as having the same properties in all directions whether in tension or compression and, furthermore, that there is no permanent deformation of the volume of a paper sheet as a result of the calendering process. As I have noted in an earlier work *(7)*, this is not a good model for paper deformation in a calender nip, because paper is a compressible, porous material that experiences permanent volume deformation as a result of the calendering process.

While the theory of plasticity can get complicated, the reader should not become distracted by discussions of tensor quantities and yield conditions. The fundamental issue is this: Pfeiffer's theory implies that all permanent deformation of paper going through a calender nip is in shear deformation (change of shape) and that there is no permanent change in volume. I contend that paper undergoes a permanent change in volume as it passes through a calender nip, and that this permanent change in volume is not negligible in comparison with the shear deformation.

This controversy should be simple to resolve by directly appealing to experimental measurements. If there are no permanent changes in the volume of paper going through a calender nip, then Pfeiffer's theory holds. If there are permanent volume changes, then Pfeiffer's theory is incorrect.

Pfeiffer admits that there are permanent changes in the thickness of the paper as it travels through the calender nip. However, his model proposes that paper behaves just as a plastically incompressible material going through a calender, with no permanent changes in volume. Pfeiffer's theory implies that there have to be permanent elongations in the machine direction (MD) as well as the cross-machine direction (CD) to compensate for the permanent contraction in the thickness direction (ZD). This means that the sums of these elongations (in percentages) have to equal the (percent) change in thickness.

For illustration purposes: if there is a 20% permanent change in thickness, then, according to Pfeiffer's theory, there should be a 10% elongation for each of the CD and MD directions, so that the volume of the paper is conserved. This would mean that for a machine with a paper width of 7.62 meters (300 in.), if the thickness experiences a 20% change when going through the calender, the paper would have to have been permanently elongated by 0.762 meters (30 in.) in the CD direction. This kind of permanent change in CD width should be easy to check!

There are a variety of methods that could be used to determine whether or not calendering induces a permanent change in volume. One could take microscopic pictures of the cross section of paper, or one could check to see whether there is a density variation as a result of the calendering process.

Thought experiments

To explain the reasoning behind his theory, Pfeiffer has offered a number of thought experiments involving familiar materials. For example, he discussed the plastic deformation of putty *(22, 23)* and, more recently, he discussed butter and bread dough as models for paper *(10)*. The problem with these thought experiments is that putty, butter, and dough are not good models for paper because they lack a vital ingredient: a large amount of air (as a percentage of its total volume) and hence the cellular character of paper. Furthermore, butter becomes a viscous liquid at (summer) ambient temperatures, and hence its behavior is more like a viscous fluid than a solid. While I do not know much about bread dough, I do not think that this is a good material model for paper.

Seth wrote a paper *(24)* wherein he advocates the use of a hydrostatic-pressure-dependent yield condition (rather than the von Mises yield condition) to describe bread dough's plastic behavior, because the more hydrostatic pressure one applies to it (while keeping a constant shear), the less permanent deformation it will experience. Better models *(7)* for paper are materials like bread slices (after rather than before baking), honeycomb, foams, etc., which are cellular and hence can experience a considerable amount of volume compressibility.

Pressure-dependent yield conditions

Hydrostatic-pressure-dependent criteria were developed very early on to analyze the elastic limit of materials. For example, see the original paper by Coulomb *(25)*, dated more than 200 years ago, where he discussed the elastic limit of soil materials. Coulomb realized that the plastic deformation of soils is influenced not only by (the maximum value of) the principal shear stress but also by the hydrostatic pressure. Coulomb also is known for working on the friction law that bears his name. He reasoned that a granular material like sand derives all its cohesion from hydrostatic pressure pushing the grains together, so that one has to overcome the friction between the grains in order to be able to slide (shear) the grains by each other. Conversely, he reasoned that such a granular material would fail if one were to apply a hydrostatic tension, since the cohesion between the grains would be lost. Thus the sign of the hydrostatic stress (whether tensile or compressive) plays a big role in this criterion, with tensile hydrostatic stresses accelerating elastic failure and compressive hydrostatic pressures delaying elastic failure.

More recently, Mohr *(18, 19)* further elaborated this theory by applying his "Mohr circle" formalism to it and extending it to the three-dimensional case. This criterion (dependent on the maximum principal shear stress and the hydrostatic pressure) is known as the Mohr–Coulomb criterion, and one can find it in a number of textbooks *(14)*.

Schleicher *(26)* considers the total strain energy as a function of the hydrostatic stress. Schleicher's yield condition depends on the value of the elastic Poisson's ratio (because he considered the total strain energy rather than just the distortional strain energy) as well as the value of a couple of constants (plasticity due to distortion and volume change).

It is interesting that von Mises himself employed a hydrostatic-pressure-dependent criterion in a paper *(27)* he wrote 36 years after his seminal paper, while he was a professor at Harvard University. Here, von Mises included square as well as fourth-power terms of the hydrostatic pressure in addition to the square of the principal shear stress [terms present in his original criterion *(16)*]. However, his motivation was to simplify the mathematical solution of a particular plane-stress rigid-plastic problem, and he does not seem to have been guided by a physical motivation. (The original von Mises *(16)* criterion results in regions where the boundary-value problem is hyperbolic and regions where the boundary-value problem is elliptic. His new hydrostatic-pressure-dependent criterion *(27)* results in a boundary-value problem that is hyperbolic in the entire region.) Based on this development, von Mises' new yield criterion *(27)* is dependent only on the magnitude of the hydrostatic stress and not on the tensile or compressive nature of that stress.

Drucker and Prager *(28)* extended von Mises' original criterion *(16)* by adding a linear term, proportional to the hydrostatic pressure, to the original von Mises' expression (the square root of the sum of the squares of the principal shear stresses). The motivation of Drucker and Prager was to model soils, and they were guided by the Mohr–Coulomb criterion, which is dependent on the sign of the hydrostatic pressure, as noted previously. It is interesting that one can interpret this yield criterion to be a modification of Schleicher's yield criterion by considering the distortional strain energy (instead of the total) to be a function of the hydrostatic pressure.

Nowadays, the preferred plastic-yield criterion to model soils is probably the Cam–Clay model *(29)*. For plane stress, the Cam–Clay criterion reduces to an ellipse in a plane defined with the hydrostatic stress and the von Mises stress as axes.

There are a number of good papers on the mathematical modeling of compaction and forging of porous metals *(30–40)*. The powder-metallurgy forming process involves the compaction of a "preform" of metal powder, with 10–30% porosity (defined by the ratio of void volume to total volume), sintering (forming a homogeneous mass by heating) the preform, and pressing it to reduce the porosity to less than 1%. Forging results in a large amount of anisotropy because of the collapse of pores and their alignment in a particular direction. On the other hand, "upsetting" a preform in a die results in a final piece that is much more isotropic than one made by forging. Kobayashi and Im *(38–40)* use a plasticity formulation that extends von Mises' *(16)* original yield condition by adding a term proportional to the square of the hydrostatic pressure. It is interesting that one can interpret this yield criterion to be a modification of Beltrami's criterion by considering the Poisson's ratio in the strain energy to be an arbitrary plastic property.

A recently published book *(41)* deals exclusively with the subject of cellular materials. Gibson and Ashby consider paper to be a cellular material. Other fabricated and natural materials considered to be cellular are foams, honeycomb, felts, cotton wool, meringue, bread (expanded by fermentation of yeast as a blowing agent), corn flakes, cork, balsa wood, sponge, cancellous bone, and coral. Although most of the book deals with simple mechanical models for cellular materials, in terms of their porosity and their cell properties, the authors briefly consider a multiaxial plastic-yield criterion. They extend von Mises' yield criterion by adding a term proportional to the square of the hydrostatic pressure. This Gibson–Ashby criterion does not compare well with experimental results. They also offer a "brittle failure" criterion, which takes into account crushing of the cells in the cellular material. This crushing criterion is essentially the Drucker–Prager criterion with particular material constants. However, I think that Gibson and Ashby's proposed criteria (with their proposed material properties) do not match experiments done for cellular materials [e.g., experiments by DeRuntz *(42)*] as well as the Cam–Clay model *(29)* could do with a good choice of material properties.

DeRuntz *(42)* conducted research on foams, made by a combination of resin and hollow glass microspheres, containing air voids. He and Hoffman performed triaxial testing of these foams by a combination of hydrostatic and uniaxial loads. The experiments showed a failure envelope that can be approximately described as an ellipse in a plane defined with the von Mises stress and the hydrostatic pressure as axes. Hence, a failure surface like the Cam–Clay model should be able to do a decent job in modeling such foams.

Anisotropic yield conditions

Paper is a material that has very different elastic properties in different material directions, the usual principal material directions being the MD, CD, and ZD. The technical word to describe this material behavior is "anisotropy" or the more correct but less commonly used term "aeolotropy" ("like the wind," i.e., with different properties in different directions). The nonlinear properties of paper are less well known (especially in the ZD). It should not be surprising to find that the nonlinear material properties of paper are anisotropic as well, since even materials that are isotropic to start with become anisotropic with plastic deformation. Such anisotropy occurs by movement of dislocations in preferential directions for crystalline metals, or by collapse of and elongation of cells in preferential directions for cellular materials.

One of the earliest proposals for an anisotropic plastic-yield criterion was advanced, in another brilliant paper, by von Mises *(43)* 15 years after his seminal paper on plastic yielding of isotropic materials *(16)*. He proposed a general quadratic expression, which he specialized for the requirement of independence of hydrostatic stresses. Hill *(15, 44)* further specialized von Mises' anisotropic yield criteria and obtained solutions *(15)* to some anisotropic problems (plane indentation of an anisotropic material by a rigid die). Hill assumed that yielding could be expressed by a function of the squares of the shear stresses and the squares of the differences of the normal stresses. As such, it was not affected by hydrostatic pressure or by the sign of the stress (i.e., it made no difference whether the stress was tensile or compressive). His criterion reduces, for vanishingly small anisotropy, to von Mises' *(16)* original criterion for an isotropic material.

The proper way to analyze the plastic deformation of paper going through a calender nip is to use a more general criterion. Such a criterion should include linear as well as quadratic stress terms, and it should take into account hydrostatic pressures as well as differences between tension and compression. My preference is for the type of criterion first offered by Gol'denblat and Kopnov, a version of which, for a particular choice of exponents, is known in the United States as the Tsai–Wu criterion *(45)*, and is commonly used to express a failure envelope for composite materials. Gol'denblat and Kopnov gave a very general expression *(46, 47)*. A particular case of it was compared with experiments on glass-reinforced composite materials, and the model successfully matched the experimental data *(48–51)*.

Although some of these criteria were initially advanced as failure conditions rather than plastic-yield criteria, one can derive stress–strain relations for the plastic state (relating the stresses to the plastic strain rates) by assuming that the plastic strain rates are given by the gradient of the yield function, as done initially by von Mises *(43)* and later by Hill *(44)*.

This report will not explore this subject further because the focus here is on the importance of volume compressibility in analyzing the behavior of paper in a calender nip. In addition, the triaxial, nonlinear, anisotropic properties of paper are poorly known. [There are some anisotropic strength studies of paper, but they confine themselves to the in-plane biaxial (MD–CD) properties of paper or paperboard *(52–54)*, and they do not deal with the thickness properties, which are of paramount importance in the calendering process.] Furthermore, replication theory as exposed by Pfeiffer *(2, 10)* is predicated on the von Mises yield criterion *(16)*, which assumes isotropy of the most general kind as well as hydrostatic stress independence. Because paper is stiffer in the MD and CD than in the ZD (the result of preferential alignment of fibers in the plane of the sheet), consideration of paper anisotropy will lead to even larger compressibility and volume deformation than an assumption of isotropy. (It is easy to see that the larger the MD and CD stiffnesses are, as compared with the stiffness in the ZD, the less the sheet will expand laterally due to pressure applied in the ZD.)

Also, as previously remarked *(7)*, plastic yielding and the subsequent response of paper are dependent on the time rate of deformation as well as temperature and humidity. Therefore, the material properties present in the plastic-yield condition should be evaluated at the proper deformation rate, temperature, and humidity for the problem one is analyzing. (If one feels that the problem at hand requires a more careful analysis of these variables, one could use a strain-rate-dependent yield function, as I did *(55)* while studying large, transient, plastic deformations. The finite-element calculations were comparable with data from experiments performed on impacted and explosively loaded metals.)

Finite-element analysis of paper in a calender nip

I conducted finite-element analyses of a paper sheet in a calender nip to obtain the (elastic–plastic) stress and strain distribution within a paper sheet as a result of calendering. These analyses took full account of geometrical as well as material nonlinearities. Nonlinearities are present because of:

- The contact problem. The contact area is an unknown, and iteration must be performed to determine the contact area at a given loading.
- Large strains. Since the calendering process involves a large contraction of the paper sheet thickness, the usual small-strain assumptions of the linear theory are not applicable.
- Elastic–plastic material behavior, including strain-hardening.

Material formulation

Finite-element analyses were carried out assuming isotropic, elastic–plastic, strain-hardening material behavior. Calculations were done for two sets of yield conditions (with associated flow rules):

1. von Mises yield condition (no permanent volume deformation)
2. Drucker-Prager yield condition (permanent volume deformation).

The Drucker-Prager yield condition, with a negative "angle of friction" *(28)*, is the simplest extension of the von Mises yield condition that includes permanent volume deformation. This yield condition was chosen for the sake of comparison with the von Mises yield condition advocated by Pfeiffer *(2, 10)*.

Finite-element formulation

Finite-element formulations were carried out using assumed-displacement, isoparametric, eight-noded, plane-strain finite elements. The number of finite elements to model paper was 2700.

The plane state of total strain in the CD is defined in Eq. 1.

$$\varepsilon_{CD} = \varepsilon_{e,CD} + \varepsilon_{p,CD} = 0 \qquad (1)$$

where

ε_{CD} =total strain rate in CD

$\varepsilon_{e,CD}$ =elastic strain rate component

$\varepsilon_{p,CD}$ =plastic strain rate component.

The elastic and plastic components in the CD, $\varepsilon_{e,CD} = -\varepsilon_{p,CD}$, generally are not equal to zero.

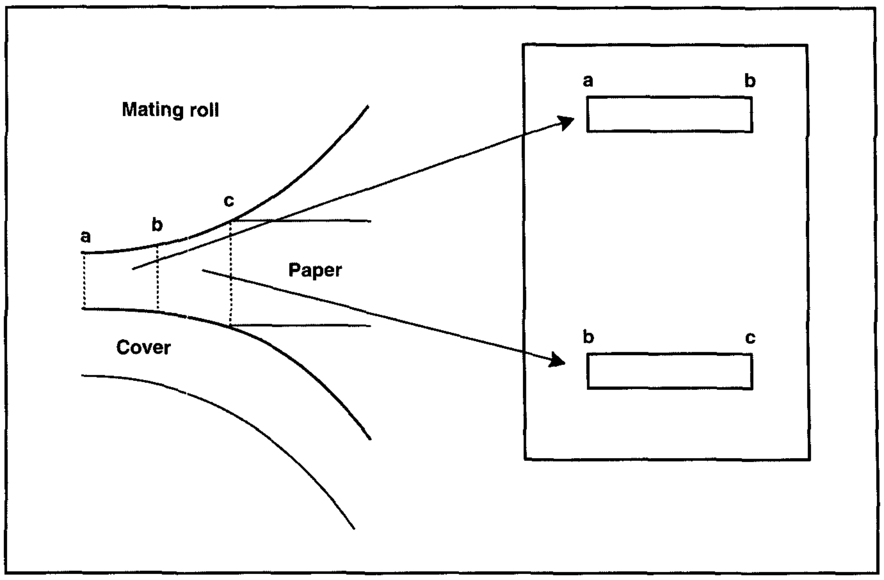
Calculations also were carried out under the assumption of static loading with sticking conditions. Tractive rolling would require the consideration of anisotropic hardening to take into account possible elastic unloading and reverse plastic loading. Therefore, tractive rolling was not modeled because, in addition to anisotropic hardening, it would be consistent to also include the initial elastic and plastic anisotropy of paper, which are excluded for previously stated reasons. The purpose of the model considered in this work is to show the importance of hydrostatic stresses.

Finite-element analysis was carried out using a total Lagrangian (second Piola–Kirchhoff stress/Green–Saint–Venant strain) formulation with a full Newton–Raphson iteration (stiffness matrix fully decomposed at every increment). The convergence check—required to establish convergence at every increment—was based on a force norm of 0.0001, a displacement norm of 0.0001, and an energy norm of 0.0001. The number of increments (of unequal length) of nonlinear solution was 170. The actual number of iterations per increment required for convergence at any given increment ranged from a minimum of two to a maximum of five.

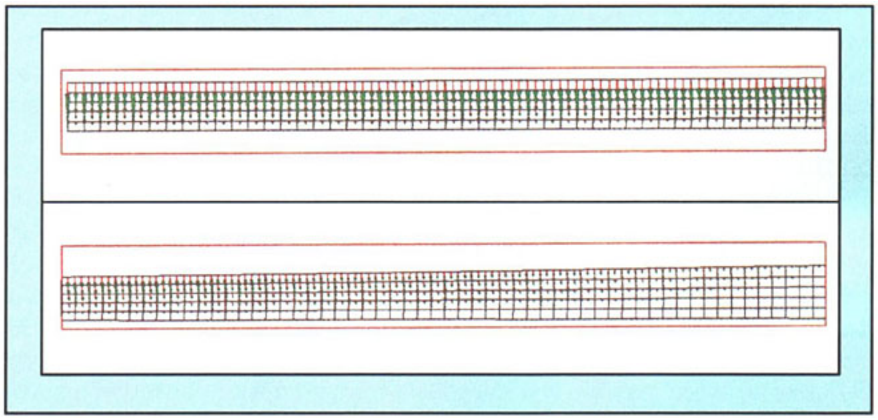
Calender parameters

The assumed line load ranged from zero to 350 kN/m (zero to 2000 lb/in.).

1. Location of paper sections illustrated in Figs. 2, 3, 5, 6, and 11. (Figure is not drawn to scale.)



2. Displacement vectors in the MD–ZD plane, von Mises yield condition. Load = 350 kN/m (2000 pli); maximum deflection = 0.033 mm (0.0013 in.).



The diameter of the covered roll was 0.599 m (23.6 in.). The diameter of the mating (metal) roll was 0.627 m (24.7 in.). Cover thickness was 0.0127 m (0.5 in.), and paper thickness before calendering was 0.127 mm (0.005 in.).

Material properties of paper

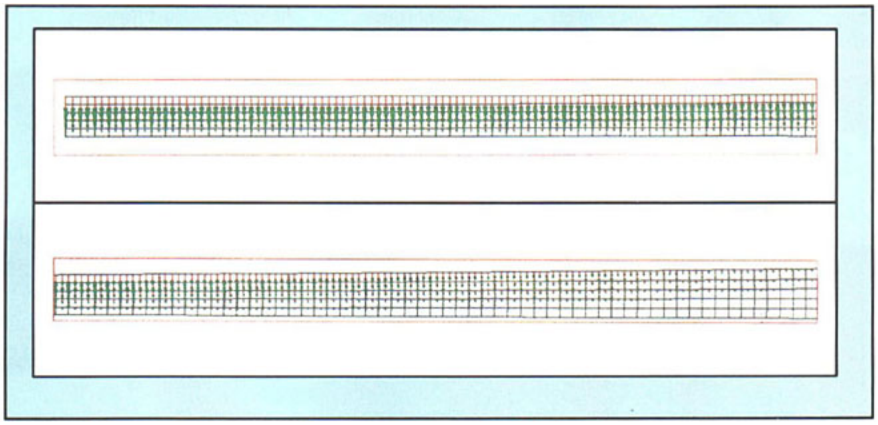
The elastic Poisson's ratio for the paper was $\upsilon = 0.05$. Young's modulus of elasticity was $E = 690$ MPa (100,000 psi). The strain-hardening tangent-modulus was $E_T = 0.25\,E$. The uniaxial yield stress was $\sigma_0 = 6.90$ MPa (1000 psi). The Drucker–Prager mean-stress coefficient was $A = -0.8571$, and the Drucker–Prager yield-stress coefficient was $B = 0.7423\sigma_0$

The choice of $A = -0.8571$ and $B = 0.7423\sigma_0$ is equivalent to an "angle of friction" of −30° and a "cohesion" of $0.5\sigma_0$ in the terminology usually used with the Drucker–Prager yield condition. Alternatively, one can get a physical picture of what these values mean by noting that the mean-stress coefficient A gives the purely plastic Poisson's ratio as

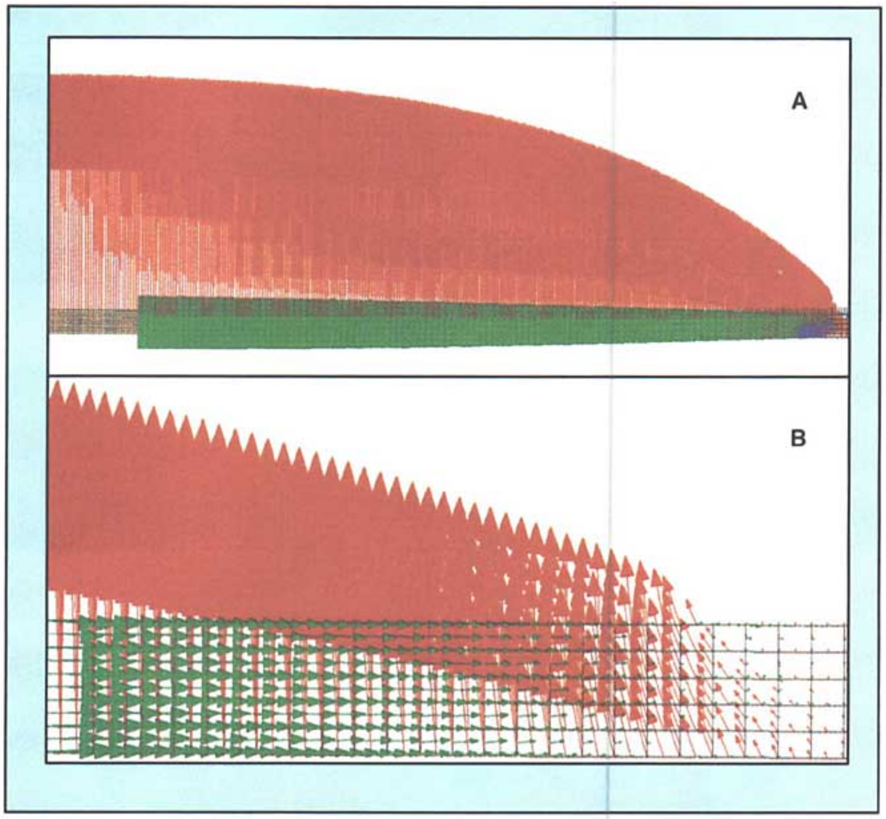
$$\upsilon_p = [0.5 + (A/3)]/[1 - (A/3)] \qquad (2)$$

Therefore, this choice of $A = -0.8571$ is equivalent to a purely plastic Poisson's ratio of $\upsilon_p = 0.167$ for the Drucker–Prager yield condition. Con-

3 Displacement vectors in the MD–ZD plane, Drucker–Prager yield condition. Load =350 kN/m (2000 pli); maximum deflection = 0.038 mm (0.0015 in.).



4. Principal (Cauchy) stress vectors in the MD–ZD plane, Drucker–Prager yield condition. Load = 350 kN/m (2000 pli). A: entire area of contact shown in Fig. 1; B: edge of contact.



versely, for the von Mises assumption, $A = 0$, and therefore $\upsilon_p = 0.5$, which implies plastic incompressibility.

Yet another physical picture of the choice of $A = -0.8571$ and $B = 0.7423\sigma_0$ can be obtained by considering the porous-metal-type of yield condition derived by Cocks [*(56)*, Eq. 28] for a full range of triaxial stresses and void-volume fractions. If one matches the Drucker–Prager yield condition (using $A = -0.8571$ and $B = 0.7423\sigma_0$) with the yield condition derived by Cocks, one concludes that the values for A and B are equivalent to a volume fraction of voids of $\rho = 0.20$.

Material properties of cover

The Young's modulus of elasticity for the cover was 1379 MPa (200,000 psi) at operating temperature. The Poisson's ratio of the cover was 0.38. With this value of Poisson's ratio, the covered roll experiences no hoop strains or peripheral speed difference with respect to the mating roll *(8)*.

Discussion

Because the contact area is much larger than the sheet thickness and because of the advanced state of plasticity (the whole sheet thickness is in the plastic state), there is practically no variation through the thickness of all stress and plastic strain components, with the exception of the shear stress and the plastic shear strain. Also, the von Mises and Drucker–Prager stresses exhibit little variation through the thickness. Large variations in the MD occur mainly at the edge of contact. The hydrostatic (mean) stress is larger than the von Mises stress for most of the contact area, except at the edge of contact.

Figure 1 is a diagram of the contact problem in the MD–ZD plane. This figure indicates the location of the exploded views of the paper's displacement vector fields and the contour plots presented in subsequent figures.

Figures 2 and 3 show the exploded views of the displacement vector fields calculated for the two yield conditions. Figure 2 shows the results obtained using the von Mises yield condition, which does not take into account any permanent change in paper volume. Figure 3 shows the results obtained using the Drucker–Prager yield condition, which does account for a permanent change in paper volume. The displacement field is overwhelmingly in the load direction (through the thickness), so it is clear that the shear strains are not major.

Figure 4 shows the principal (Cauchy) stresses (as vectors in the MD–ZD plane) in a sheet entering the calender nip for the formulation using the Drucker–Prager yield condition. As seen in Fig. 4A, the principal stresses are predominantly oriented in the ZD (major principal stress) and the MD. The distribution is nearly

Hertzian (semi-elliptical). Only at the edge of contact (Fig. 4B) does the direction of the principal stresses change by a significant amount.

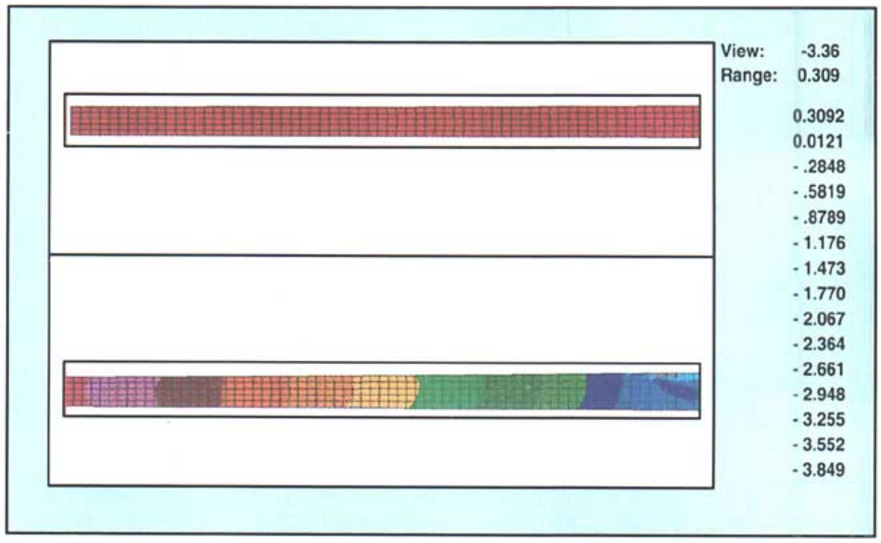
Figures 5 and 6 are contour plots showing the distribution of the ratio of the hydrostatic (mean) stress to the von Mises stress. Figure 5 shows the results under the von Mises yield condition, while Fig. 6 shows the results under the Drucker–Prager yield condition.

Figure 7 shows the ratio of the hydrostatic (mean) stress to the von Mises ("effective") stress, at the center of contact, vs. the line load. Because of the plastic incompressibility inherent in the von Mises model, the hydrostatic stress reaches large values (more than three times larger than the von Mises stress), even though the present model considers a very elastically compressible material model (elastic Poisson's ratio of 0.05). The Drucker–Prager yield condition results in smaller hydrostatic stresses (of magnitude similar to the von Mises stress) because the hydrostatic pressures can be relieved by plastic volume change in the Drucker–Prager model.

Figure 8 shows the stress in the ZD (normalized by the yield stress), at the center of contact, vs. the line load. The difference between the von Mises formulation and the Drucker–Prager formulation is not large. At the same line load, the Drucker–Prager formulation results in a somewhat smaller stress in the ZD because the von Mises formulation (due to its plastic incompressibility condition) results in a somewhat smaller contact area and hence a larger stress.

Figure 9 shows the engineering plastic strain in the ZD vs. the line load. As we would expect, the Drucker–Prager yield condition results in significantly larger strains, at the same line load, than the von Mises yield condition. The reason for this is that the von Mises yield condition assumes plastic incompressibility, while the Drucker–Prager yield condition allows for plastic volume changes.

5. Ratio of mean stress to von Mises stress in the MD–ZD plane, von Mises yield condition. Load = 350 kN/m (2000 pli).



6. Ratio of mean stress to von Mises stress in the MD–ZD plane, Drucker–Prager yield condition. Load = 350 kN/m (2000 pli).

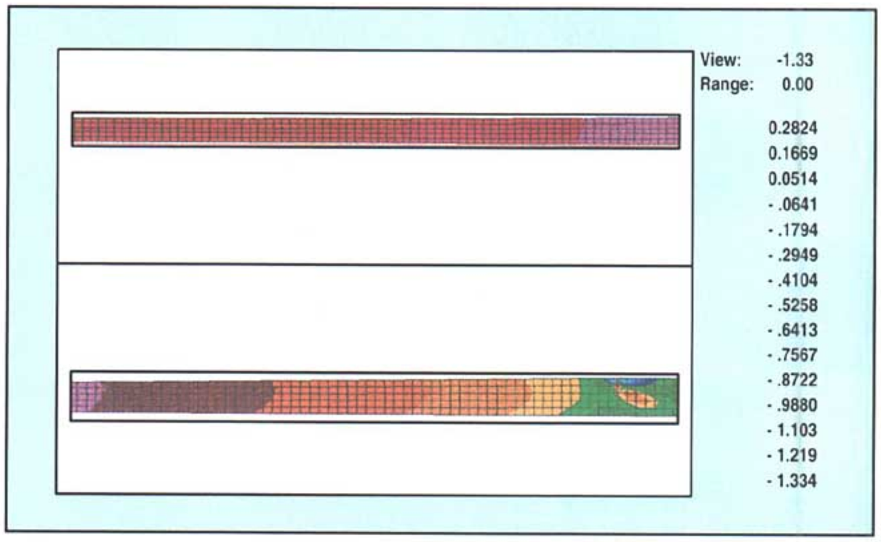
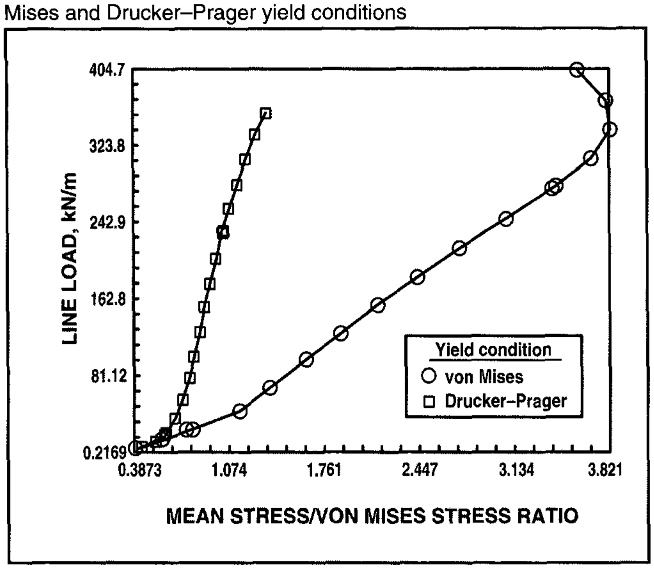


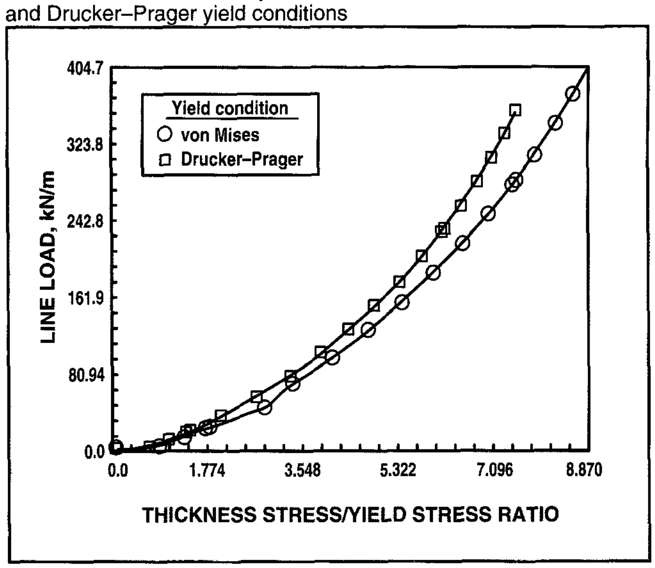
Figure 10 shows the engineering plastic strain in the CD vs. the line load. The permanent strain in the CD is much larger for the von Mises yield condition. The plastic incompressibility assumption inherent in the von Mises yield condition leads to a permanent stretch in the CD that is just about half of the permanent compression in the ZD.

While these finite-element computations were made with an assumption that the elastic Poisson's ratio is $\upsilon = 0.05$ for both the von Mises and the Drucker–Prager formulations, the plastic Poisson's ratios are $\upsilon_p = 0.5$ for the von Mises yield condition and $\upsilon_p = 0.167$ for the Drucker–Prager formulation. A given elastic–plastic state (at a given line load) will result in a given ratio of elastic strain to plastic strain, and this total state, in turn, will result in an elastic–plastic Poisson's ratio that is intermediate between the elastic and plastic values of the Poisson's ratio. Initially, the elastic–plastic Poisson's ratio is equal to the elastic value, since for very small val-

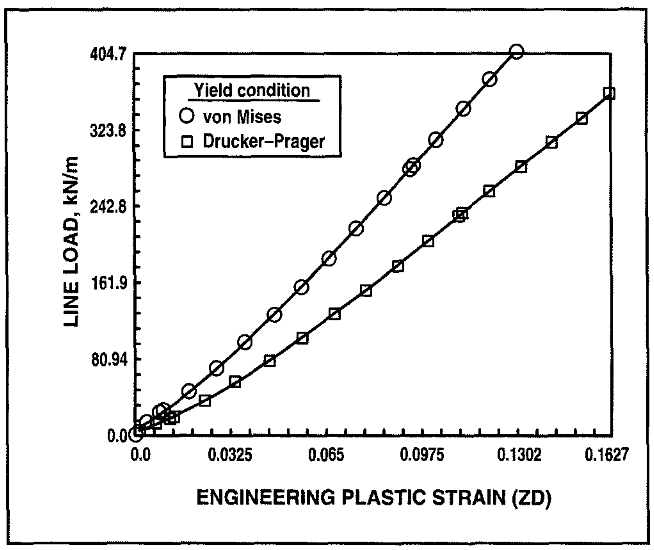
7. Ratio of mean stress/von Mises stress vs. line load for both von Mises and Drucker–Prager yield conditions



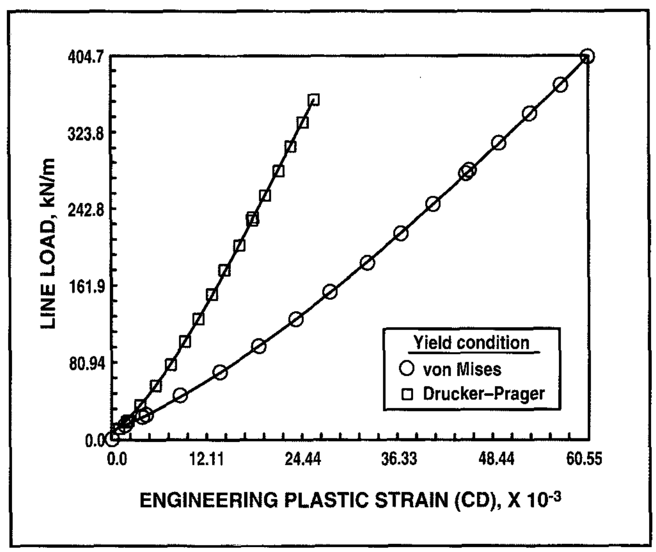
8. Ratio of thickness stress/yield stress vs. line load for both von Mises and Drucker–Prager yield conditions



9. Engineering plastic strain in the thickness direction vs. line load for both von Mises and Drucker–Prager yield conditions



10. Engineering plastic strain in the CD vs. line load for both von Mises and Drucker–Prager yield conditions



ues of the line load, the state is fully elastic. As the line load is increased, plastic strains begin to appear, and the elastic–plastic Poisson's ratio increases. Eventually, when the plastic strain is much larger than the elastic strain, the elastic–plastic Poisson's ratio reaches the plastic value. Strain-hardening delays the approach to the fully plastic value, since even at large strains there is a significant component of elasticity. This is so because strain-hardening allows the elastic component of strain to continue to increase even beyond the yield point.

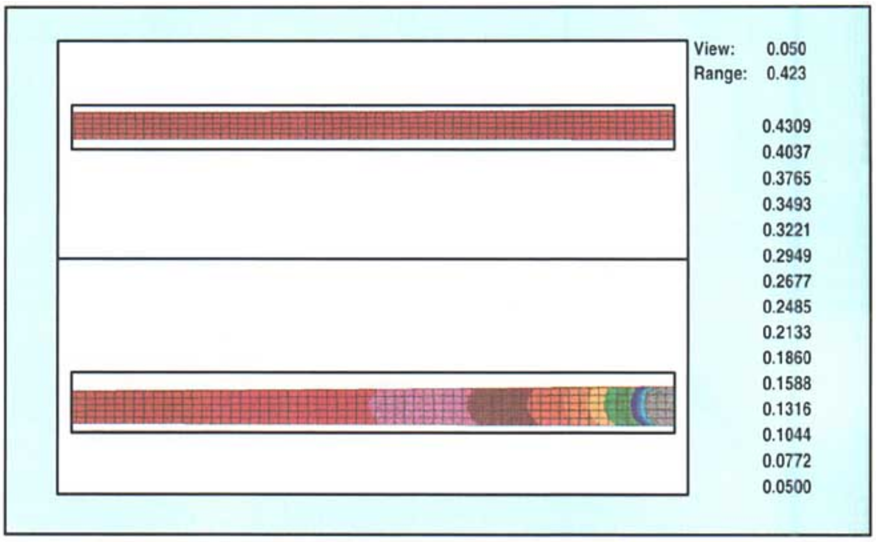
Figure 11 shows the distribution of the elastic–plastic Poisson's ratio, from the center of contact to the edge of contact, for the von Mises yield condition formulation. **Figure 12** shows the elastic–plastic Poisson's ratio, at the center of contact, vs. line load. The von Mises yield condition results in large values of elastic–plastic Poisson's ratio, approaching 0.5, due to the incompressibility assumption inherent in this condition. In contrast, the Drucker–Prager yield condition results in smaller values of elastic–plastic Poisson's ratio. These values are dependent on the values selected for the Drucker–Prager material constants. One could experimentally check whether or not plastic compression of paper results in effective Poisson's ratio approaching 0.5, and therefore decide on the suitability of the von Mises yield condition to model paper.

Finally, **Fig. 13** shows the permanent plastic volume change per unit original volume vs. line load. The permanent change in volume, at 350 kN/m (2000 pli), is about 12% according to the Drucker–Prager criterion, which takes into account hydrostatic stresses. Of course, there is no permanent change in volume with the von Mises criterion.

The small value of elastic Poisson's ratio (υ = 0.05) used in these calculations results in large elastic compressibility of the paper sheet (at the expense of plastic compressibility). This gives the benefit of the doubt to the comparison between models that do and do not consider plastic compressibility. (As previously remarked, a contact problem on a material body with a high value of Poisson's ratio results in regions that are relatively rigid.) However, even considering this small value of Poisson's ratio, the hydrostatic stresses present in the von Mises model are quite large, and considerable permanent changes in volume do occur if one uses a yield criterion (Drucker–Prager) that accounts for them.

While these finite-element analyses took into account hydrostatic stresses and permanent deformation of the volume of the paper sheet, they did not take into account the temperature gradients and/or moisture gradients in the paper sheet. These temperature gradients influence the amount of plastic deformation that occurs through the thickness of the paper sheet, because the yield stress is a function of temperature and moisture content *(7)*. (Generally, the higher the temperature, the lower is the yield stress.) In particular, it would be interesting to do a finite-element analysis of the so-called "thermal gradient calendering" *(57, 58)* and "substrata thermal molding" *(59)* processes. However, one should take into account that when the temperature of the process is high enough such that the glass transition temperature of the material is exceeded, viscoplasticity plays an important role. In this case, we will see not only elastic and permanent (plastic) deformation, but also stress relaxation. Thus the thickness and volume of the sheet may change with time after the paper exits the nip. These time effects are, of course, also expected in response to changes in moisture content.

11. Elastic – Plastic Poisson's ratio in the MD–ZD plane, von Mises yield condition. Load =350 kN/m (2000 pli).



Conclusions

Finite-element calculations show that hydrostatic stresses play an important role in the paper deformation that occurs in a calender nip.

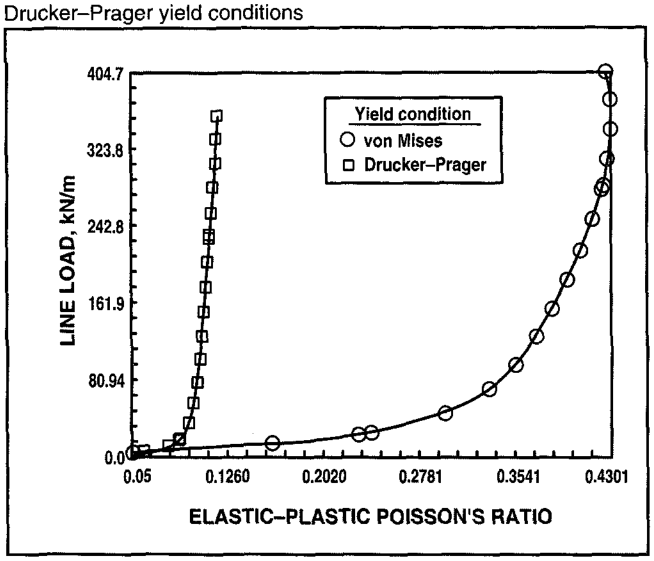
An isotropic elastic material needs two elastic constants to describe its behavior (e.g., the Young's modulus and the Poisson's ratio or the bulk modulus and the shear modulus). Similarly, an isotropic, perfectly plastic material generally needs at least two constants to describe its behavior. The reason why the von Mises yield criterion formulation needs only one constant (the yield stress) is that the von Mises yield criterion assumes plastic incompressibility. In other words, it assumes no permanent change in volume. While this condition is acceptable for most engineering materials, like metals, it should not be acceptable for cellular materials like paper. Such materials contain a large volume occupied by voids, which are compressible. Hence a model of calendering of paper should include permanent changes in volume.

The importance of the arguments made here—that hydrostatic stresses are not negligible and that there is a permanent change in volume of paper as a result of the calendering process—is not purely of an academic nature. On the contrary, this understanding can lead to practical improvements in calendering by maximizing one kind of deformation (shear) at the expense of another (volumetric). For example, this analysis of the paper deformation in the calendering process reveals that if one wants to maximize permanent (plastic) deformation of the paper sheet, while minimizing permanent change in volume, one should:

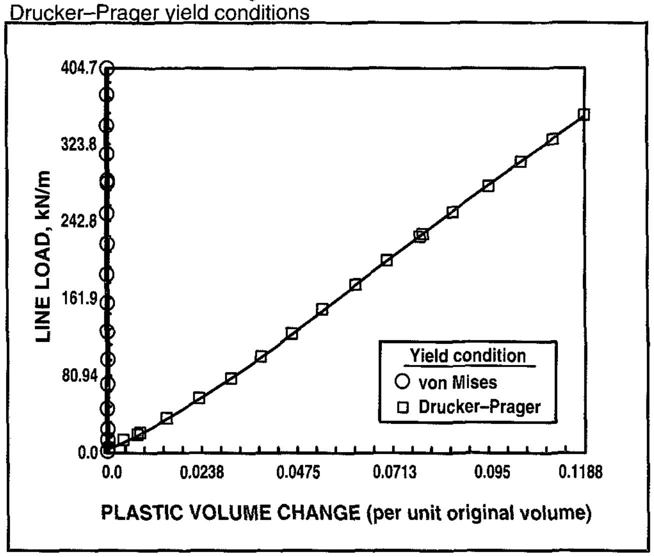
- Minimize the coefficient of friction to a value of 0.04 or less, as done in the calendering of metals *(60, 61)*, in order to allow the sheet to "squeeze out" and therefore minimize plastic volume deformation. This could be accomplished by minimizing the surface friction of the paper or coating surfaces and/or the surface friction of the covered roll and the mating roll.
- Maximize the tension on the sheet to reduce the amount of hydrostatic stress and maximize the principal shear stresses on the sheet.

It should be simple to determine whether calendering produces no permanent change in paper volume, as

12. Elastic – Plastic Poisson's ratio vs. line load for both von Mises and Drucker–Prager yield conditions



13. Plastic volume change vs. line load for both von Mises and Drucker–Prager yield conditions



predicted by the von Mises criterion advocated by Pfeiffer *(2, 10)*, or whether calendering does produce a permanent change in volume, as I advocate.

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Appendix

Young's modulus of elasticity: E
Poisson's ratio: υ
Shear modulus of elasticity: $G = E/[2(1+\upsilon)]$
Bulk modulus of elasticity: $K = E/[3(1-2\upsilon)]$
Uniaxial yield stress: σ_0
Plastic-yield properties: $A, B, H, I, M, N, P, T, X, Z_{IJ}, Z_{IJKL}, \alpha, \beta$
Principal normal stresses: $\sigma_1, \sigma_2, \sigma_3$
Principal or "maximum" shear stresses: $\tau_3 = (\sigma_1 - \sigma_2)/2$
$\tau_2 = (\sigma_1 - \sigma_3)/2$
$\tau_1 = (\sigma_2 - \sigma_3)/2$
Mean or "hydrostatic" stress: $\sigma_m = (\sigma_1 + \sigma_2 + \sigma_3)/3$
Von Mises or "effective" stress: $\sigma_e^2 = 2(\tau_3^2 + \tau_2^2 + \tau_1^2)$
Tresca stress: $\sigma_T^2 = \text{Maximum of } (2\tau_3)^2, (2\tau_2)^2, (2\tau_1)^2$
Distortion strain energy (per unit volume) of an isotropic material: $U_d = \sigma_e^2/(6G)$
Volumetric strain energy (per unit volume) of an isotropic material: $U_v = \sigma_m^2/(2K)$
Total strain energy (per unit volume) of an isotropic material: $U = U_d + U_v$
Von Mises yield condition: $\sigma_e^2 = \sigma_0^2$
Tresca yield condition: $\sigma_T^2 = \sigma_0^2$
Beltrami (strain energy) yield condition: $2EU = \sigma_0^2$
Schleicher yield condition: $2EU = (\sigma_0 - X\sigma_m)^2$
Drucker–Prager yield condition: $A\sigma_m + \sigma_e = B$
Porous-metals yield condition: $I(\sigma_m^2) + \sigma_e^2 = H$
Mohr–Coulomb yield condition: $M\sigma_m + \sigma_T = N$
Cam–Clay yield condition: $P(\sigma_m^2) + \sigma_e^2 = -T\sigma_m$
Anisotropic (Gol'denblat) yield condition: $(Z_{IJ}\,\sigma_{IJ})^{\alpha} + (Z_{IJKL}\sigma_{IJ}\sigma_{KL})^{\beta} + \ldots = 1$

The Tresca and Mohr–Coulomb criteria present the difficulty that the gradient of the yield function is not a continuous function of stress. Contrast this with the von Mises criterion, where the gradient is a continuous function of stress.