Random star patterns and paper formation

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ABSTRACT: A random fibrous network is the basic reference structure for the study of paper. Commercial papermaking, however, introduces two principal nonrandom influences: fiber orientation and flocculation. Analytical treatments have been developed to elucidate the effect of fiber orientation. This article extends the treatment to include flocculation, which is modeled as a random deposition of star-shaped clumps of fibers. The approach is based on the derivation of the point autocorrelation function for stars with different numbers of arms. For commercial newsprint, the coefficient of variation of local grammage, at the scale of one fiber length, is typically double that for random paper made from the same furnish. We show that such a level of variability arises from the deposition of stars having four arms. However, analysis of commercial fourdrinier papers shows that their variability decays less rapidly with scale than could occur if their formation were well represented by random patterns of stars. This suggests a super-floc cooperative drainage effect.

KEYWORDS: Formation, flocculation, mathematical models, fibers, random processes.

Analysis of the phenomenon of flocculation is qualitatively different from that of preferential fiber orientation *(1–4)* because it manifestly involves nonindependent events that are not easily amenable to superposition procedures. These events themselves appear to be an intricate mix of stochastic and deterministic processes that are not well understood.

The situation is changing, however, because it is now possible to measure small-scale variations in fiber velocities at the slice and on the wire. Indeed, Loewen and Buttler *(5)* used infrared laser-Doppler anemometry to measure remotely the velocity of the suspension on the forming table of a paper machine and showed that this technique has potential for measuring the distribution of principal fiber directions across the machine. [Cf. also Dahl *et al.* *(6)*]. Moreover, hot-film anemometry appears to be capable of providing data on local flow dynamics in the headbox, while local formation data at the dry end can be obtained optically. Both methods provide data at the scale of fiber length.

We have an analytical treatment of the three-dimensional spatial statistics of stochastic fibrous networks during their evolution from a fiber suspension to an essentially layered structure in dry paper *(7, 8)*, and Steen *(9–12)* has a physical approach to the linking of flocculation and turbulence. Thus, we can model various possible ways in which the three-dimensional formation index evolves during the papermaking process, and the ensuing histories can be classified. Our objective is to improve monitoring of local formation and thus move toward its optimal control.

When discussing the variability in structure of different papers, it is convenient to normalize the measured standard deviation of local grammage, $\tilde{\beta}$, by dividing by the mean, $\bar{\beta}$, which gives the coefficient of variation, CV($\tilde{\beta}$). Another method is to use the formation number, n_f, the ratio of measured variance to the expected variance for a corresponding random fibrous network.

$$n_f(x) = \mathrm{Var}_{\mathrm{paper}}(\tilde{\beta})/\mathrm{Var}_{\mathrm{random}}(\tilde{\beta}) \qquad (1)$$

Of course, both indices vary with x, the size of the inspection zone.

We know that at the scale of about 1 mm, typical fourdrinier papers have four times the variance or, equivalently, twice the coefficient of varia-

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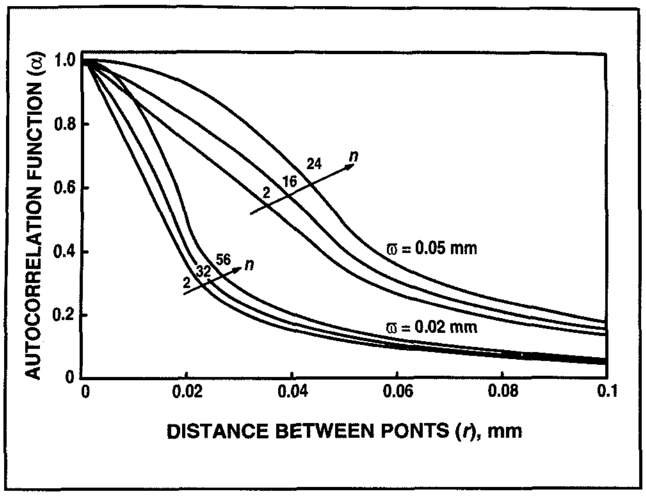
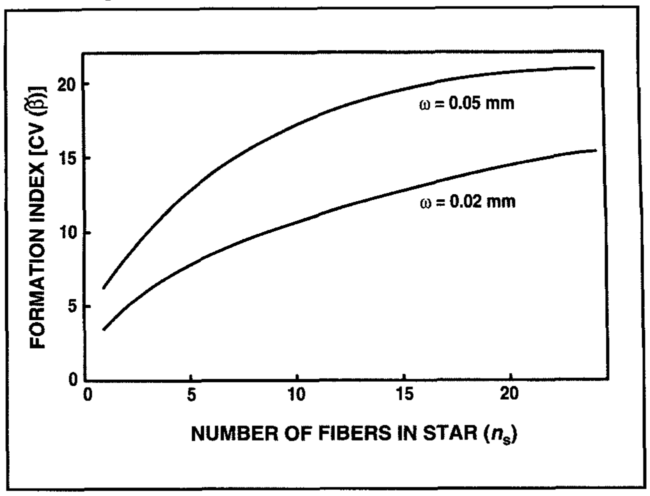
1. Point autocorrelation function as a function of distance for various levels of n_s (number of fibers within star). Fiber length λ =1 mm, and fiber width ω =0.02 mm and 0.05 mm.



2. Formation index of random star patterns as a function of n_s. Fiber length λ =1 mm; fiber width ω =0.02 mm and 0.05 mm; square zones of side length x =1 mm; mean coverage of 10 fibers (≈50 g/m²).



tion of a random paper made from the same furnish *(13)*. Thus the formation number at the scale of 1 mm, $n_f(1)$, is approximately four for a fourdrinier sheet. Also, Kiviranta and Paulapuro *(14, 15)* have shown that 10–20% of the variation in commercial fourdrinier paper can be attributed to table turbulence. Dodson and Schaffnit *(13)* found that a similar fraction could be correlated to static furnish variables.

The scale-dependent variability of commercial paper is qualitatively similar to that of random paper. This can be approximated over the normal scale range of interest by a simple model for flocculated structures with coefficient of variation

$$CV_{floc}(\beta) = n_f^{1/2}\, CV_{random}(\beta) \qquad (2)$$

$$= [n_f(x)]^{1/2}\, (\delta\tilde{\rho}/\omega)^{1/2}\, (1/\bar{\beta})^{1/2} \qquad (3)$$

It turns out that, because $\tilde{\rho}$ is approximately proportional to fiber width *(16, p. 205)*, the expression for $CV_{floc}(\bar{\beta})$ represents also the coefficient of variation for a random array of rectangular clumps of fibers. Mathematically, $n_f(x)$, if constant, represents approximately the number of fibers in a clump such that, if the clumps are dropped at random, then they would yield the same variability in structure as the flocculated case. However, the true picture is more complex, because we find that $n_f(x)$ for commercial paper increases with x, rather than remaining constant. This is consistent with the presence of a drainage effect, first proposed by Wrist *(17)*, whereby hydrodynamic forces cause fibers to deposit preferentially on lower-density regions of a forming pad.

Experimentally, $1 \leq n_f(1) \leq 7$, and $n_f(1)$ is linearly correlated with both the crowding number, n_c, and with fiber length (or freeness, °SR) in the headbox on a wide range of fourdriniers producing papers from tissue to board *(13)*. (The crowding number, n_c, is the number of fibers expected in a sphere with a diameter of one mean fiber length.) Similarly, we have simulated on a supercomputer a range of flocculation processes and found the same qualitative structural statistics.

These findings were helpful in our efforts to understand the statistical geometry of fiber flocculation analytically, and this encouraged us to generate a new family of reference structures. We observed that the variance–aperture relation is similar for commercial and random paper, with only the scale altering. This suggested that paper may be rather amorphous above the dimensions of typical flocs and, hence, that random floc deposition should by analyzed. We did this using flocs that were regular stars made from n_s uniform fibers of length λ and width ω, for a range of numbers of arms n_s.

Point autocorrelation function

The theory of complete sampling with finite cells in a random fibrous network was given in a previous work *(18)*, which showed how the point autocorrelation function, α, for single fibers was derived and integrated. Instead of individual fibers, first we considered elements consisting of just two fibers crossed centrally at an angle η and extended the analysis given in the previous work *(18)*. We derived the point autocorrelation function for different angles and found that, for our ranges of interest, it is essentially independent of the angle η. Next, for the general situation of n_s fibers clustered into a regular star, we derived the effect of the number n_s on the point autocorrelation function.

Some results are shown in **Fig. 1.** As expected, the effect of increasing the number of fibers in the stars is to make the correlation stronger and persist for longer distances.

Effect of star size on formation

For a given type and size of inspection zone, it is possible to compute the vari-

I. Grammage and headbox data for 22 uncoated fourdrinier papers.* Dimensionless indices—crowding number, n_c, and formation number, $n_f(x)$—were developed from these data.

Sample no.	Headbox data: Mean grammage (β), g/m^2	Freeness, °SR	Mean consistency (γ), %	Mean fiber length (λ), mm	Crowding no., n_c	Formation nos.: $n_f(1)$	$n_f(2)$	$n_f(3)$	$n_f(4)$	$n_f(5)$
1	16.6	16	0.21	1.095	6.73	3.79	5.71	8.58	11.83	15.54
2	23.1	44	0.24	1.209	10.26	3.72	4.77	6.98	9.12	11.32
4	40.8	85	0.52	0.858	6.26	1.02	1.86	2.64	3.63	4.24
5	41.9	30	0.41	1.256	16.44	3.15	4.09	5.83	8.02	9.44
6	42.1	28	0.30	1.999	17.93	2.57	3.52	5.91	7.98	8.94
7	42.4	85	0.52	0.858	6.26	1.26	2.28	3.33	4.26	5.20
8	50.0	15	0.80	0.798	13.01	2.45	4.74	7.05	9.51	11.90
9	50.3	65	0.46	0.656	3.96	1.68	3.28	5.59	8.08	10.17
10	55.3	38	0.32	0.748	4.13	3.23	6.33	8.91	13.74	17.20
11	58.1	48	0.33	0.802	4.63	2.45	4.54	7.44	10.75	12.15
12	75.7	53	0.58	1.012	23.74	4.03	6.44	8.14	11.12	16.35
13	76.1	68	0.98	0.520	13.09	2.25	5.18	8.48	12.67	16.38
15	93.3	37	0.81	0.719	13.21	1.13	2.17	3.49	5.10	6.85
16	124.2	18	0.77	1.257	36.61	8.76	15.77	25.10	37.91	53.33
17	126.6	23	0.89	0.753	13.91	2.64	4.30	7.16	10.52	14.32
18	130.3	26	1.13	0.643	13.98	3.42	6.96	11.19	17.64	22.51
19	139.4	13	0.47	0.988	9.93	3.31	7.23	11.95	18.09	25.03
20	143.5	40	1.07	0.683	12.45	2.39	3.82	5.43	7.92	9.93
21	156.4	37	1.17	0.649	10.04	6.27	7.56	10.10	13.57	18.65
22	215.8	35	1.18	0.832	22.16	3.77	8.55	13.98	19.81	30.65
23	246.8	42	1.35	0.580	16.29	5.82	12.43	21.31	32.08	46.63
24	261.8	24	1.15	0.656	9.89	6.24	18.59	35.60	56.09	81.93

*Data from Corte *(16, 19)*

ance, Var($\bar{\beta}$), of our random star patterns. This involves taking the expectation of the point autocorrelation function α over the probability distribution function b for distances r chosen independently and at random within the given zone. We are mainly interested in square zones of side length x. In this case, the required between-zone variance is:

$$\mathrm{Var}_x(\bar{\beta}) = \mathrm{Var}_0(\bar{\beta})\int_0^{\sqrt{2}x} \alpha(r,\omega,\lambda,n_s)\, b(r,x)\, dr \quad (4)$$

where $\mathrm{Var}_0(\tilde{\beta})$ is the total, or point, variance of the structure.

This integration has been done numerically for ranges of interest, and typical results are shown in **Fig. 2** for a mean coverage of 10 fibers, which corresponds to about 50 g/m^2 for typical fibers. We see that there is a doubling of the coefficient of variation when the number of fibers in a star increases from 1 to 4. For comparison, there is a similar effect by increasing fiber width from 0.02 mm to 0.05 mm (while keeping fiber length constant at 1 mm).

Further work studied the effects of distributions of different stars in a network, and a proportional mixing effect was found. Now, the important consequence of a random disposition of any distribution of star types is to make the formation number of the resulting structure constant with inspection-zone size, whereas the values found for paper increase with increasing size of the inspection zone. This discrepancy may be due to the stars being "hard-centered," whereas real flocs are more homogeneous agglomerations.

Formation of paper

From the data of Corte *(16*, pp. 212–18*)*—used in previous work *(13)* and summarized for different inspection-zone sizes in **Table I**—we computed log-linear regression equations for $n_f(x)$, with x =1 mm, 2 mm, 3 mm, 4 mm, and 5 mm. The formation number $n_f(x)$ was found to be log-linearly correlated with the crowding number n_c

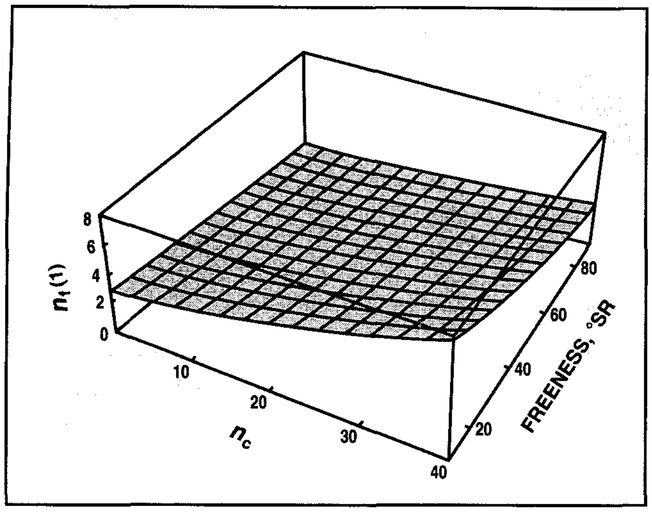
$$n_c = [\pi\gamma\lambda^2]/6\delta \quad (5)$$

and with fiber length or headbox freeness (°SR). The regression equations are significant at levels around 1–3% and are summarized as follows:

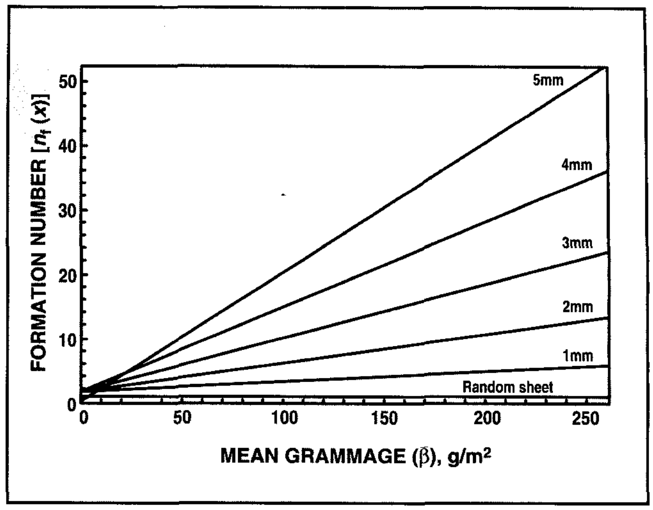
$$n_f(1) = \exp(1.26 + 0.0253 n_c - 0.0125\,°\mathrm{SR}) \quad (6)$$

$$n_f(2) = \exp(1.85 + 0.0240 n_c - 0.0123\,°\mathrm{SR}) \quad (7)$$

3. Graph of Eq. 6, the regression of $n_f(1)$ on headbox crowding number and degree of beating for fourdrinier papers.



4. Formation number as a function of mean grammage for fourdrinier paper (inspection-zone sizes 1–5 mm) and for a randomly formed handsheet



$$n_f(3) = \exp(2.36 + 0.0227n_c - 0.0136°SR) \quad (8)$$

$$n_f(4) = \exp(2.76 + 0.0226n_c - 0.0146°SR) \quad (9)$$

$$n_f(5) = \exp(2.99 + 0.0273n_c - 0.0153°SR) \quad (10)$$

We should view these as first nonlinear approximations, at the different scales, of the dependence of formation quality on headbox crowding number and beating. Equation 6, the regression equation for an inspection zone of 1 mm, is illustrated in **Fig. 3**.

Similarly, substituting mean fiber length for freeness (°SR) in the joint regression, we obtain the following equations, with significance levels again around 1-3%:

$$n_f(1) = \exp(0.721 + 0.0396n_c - 0.161\lambda) \quad (11)$$

$$n_f(2) = \exp(1.59 + 0.0440n_c - 0.544\lambda) \quad (12)$$

$$n_f(3) = \exp(2.05 + 0.0444n_c - 0.572\lambda) \quad (13)$$

$$n_f(4) = \exp(2.46 + 0.0466n_c - 0.666\lambda) \quad (14)$$

$$n_f(5) = \exp(2.77 + 0.0544n_c - 0.823\lambda) \quad (15)$$

Estimation of relative changes in n_f values is simple from the log-linear regression equations since, for example,

$$[1/n_f(1)]\,[dn_f(1)/d\lambda] = d\log[n_f(1)]/d\lambda = 0.0396(dn_c/d\lambda) - 0.16 \quad (16)$$

We also know that

$$(dn_c/d\lambda) = [d(\pi\lambda^2\gamma)/(6\delta)]/d\lambda = 2n_c/\lambda \quad (17)$$

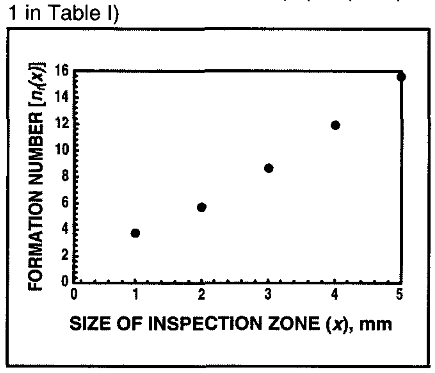
Also, relative changes in coefficients of variation of local grammage depend on relative changes in the square root of the formation number, and so we can use

$$\{1/[n_f(1)]^{1/2}\}\,\{d[n_f(1)]^{1/2}/d\lambda\} = d\log[n_f(1)]^{1/2}/d\lambda = 0.5\{d\log[n_f(1)]/d\lambda\} \quad (18)$$

We examined the relationship between n_f and mean grammage at different inspection-zone sizes (x =1, 2, 3, 4, 5 mm). The regression lines are shown in **Fig. 4**, together with the horizontal line at $n_f = 1$, which corresponds to a random sheet.

The relationship between $n_f(x)$ and inspection-zone sizes (x =1, 2, 3, 4, 5 mm) is shown in **Fig. 5** for typical lightweight (<51 g/m²) fourdrinier papers. Similar relationships are found for other papers and for handsheets *(20)*.

5. Formation number as a function of inspection-zone size for fourdrinier paper (Sample 1 in Table I)



We see from the data on fourdrinier papers that experimental values of n_f increase with mean grammage (Fig. 4) and with the size of the inspection zone within which variance is measured (Fig. 5). Random structures of single fibers, or of any star-shaped clumps of fibers representing random floc patterns, would all have n_f constant. An approach to the interpretation of this effect will be presented in future reports *(21, 22)*.

Conclusion

The statistical geometry of random patterns of fiber clumps in the form of regular stars was analyzed by computing the effect of the number of fibers in the star on the variability in

Nomenclature

n_f = formation number
n_c = crowding number
n_s = number of fibers in star
x = size of the inspection zone
r = distance between two random points within inspection zone
$b(r)$ = probability density function for distance r
$\tilde{\beta}$ = local grammage
$\bar{\beta}$ = mean grammage
$CV(\tilde{\beta})$ = coefficient of variation of local grammage
$Var(\tilde{\beta})$ = variance of local grammage
$\tilde{\rho}$ = ratio of local grammage variance to that at points, i.e., $Var(\tilde{\beta})/Var(\beta)$
λ = mean fiber length
ω = fiber width
γ = mean consistency
δ = mass per unit length of fiber
α = autocorrelation function, the ratio of the variance of the number of fibers covering a pair of points (separated by distance r) and the variance of the number covering a single point.

local grammage. For fibers of length 1 mm, a random distribution of clumps of four fibers has a similar variance to that of commercial newsprint at the scale of 1 mm. Thus the formation number at that scale is $n_f(1) \approx 4$. For commercial paper, the formation number associated with an inspection zone of size x, $n_f(x)$, increases slowly with the size of the inspection zone, whereas for random patterns of stars, $n_f(x)$ is constant. It follows that commercial paper cannot be modeled by random deposition of hard-centered flocs and that there is a super-floc cooperative drainage effect. The formation number precisely measures the extent of this effect and its dependence on scale. TJ

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