Minimum variance as a benchmark for feedback control

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Minimum variance is a better benchmark than zero variance for evaluating controller performance. Control systems cannot reduce the variance in product quality below the variance inherent in the process. On the basis of minimum variance, an investment in controller maintenance can be evaluated realistically. Minimum variance control does not have to be implemented, and controller tuning parameters have no effect on the evaluation. A performance index based on minimum variance can also be calculated.

Application:

Use minimum variance, not zero variance, as a yardstick for controller performance.

INVESTING IN THE MAINTENANCE of your control system pays you back in terms of product quality by reducing product variance. How can you estimate this payback? Knowing only that the standard deviation of your product is 2% does not help. But if you knew that your process would operate with a standard deviation of 0.5%, you might decide to go ahead with the maintenance project.

To make the right decision, you need information about the current performance, but you also need a defined benchmark against which to gauge performance. One such benchmark is zero variance, which means a perfect process. With zero variance, you expect a new controller to result in a flawless process. More realistic is the yardstick of minimum variance. With minimum variance, you expect a new controller to make the porcess the best it can be. By projecting performance against minimum variance, you can decide what constitutes a reasonable return on your maintenance investment.

MINIMUM VARIANCE AS A BENCHMARK

Minimum variance is proposed as a convenient and achievable benchmark *(1)*. Using the minimum variance *benchmark* is not the same as using minimum variance *control*, however. Minimum variance control is usually too aggressive. Nevertheless, as a benchmark, minimum variance offers several advantages.

First, using the minimum variance benchmark keeps us from making the mistake of expecting too much. We can allow for limitations inherent in the process.

Second, we can calculate the minimum variance without disturbing the process. Minimum variance control does not have to be implemented to establish the benchmark *(2)*.

Third, the minimum variance benchmark is independent of any parameters related to the controller. Alternative controllers can be compared without regard to the strategy for tuning. If one controller is tuned for minimum overshoot and the other is tuned for a fast response to a setpoint, it makes no difference.

INHERENT MINIMUM VARIANCE

All processes have an inherent minimum variance *(2)*. When your controller performs near this variance lower bound, the only way to reduce the variance further is to change the process. This lower bound is imposed by the process and not the controller.

The limit arises from stochastic disturbances and from dead-time. Stochastic disturbances are caused by unmeasured feed changes, ambient temperature swings, static friction (often called valve "stiction"), and other factors not managed under the control strategy. Dead-time arises from the time needed for a parameter value to move in response to a change in conditions, for material to be transported, and so forth. In a sampled controller, such as any distributed control system, the dead-time is at least one sample interval. However, even in analog control systems, there is a finite process dead-time that leads to the characteristic minimum variance.

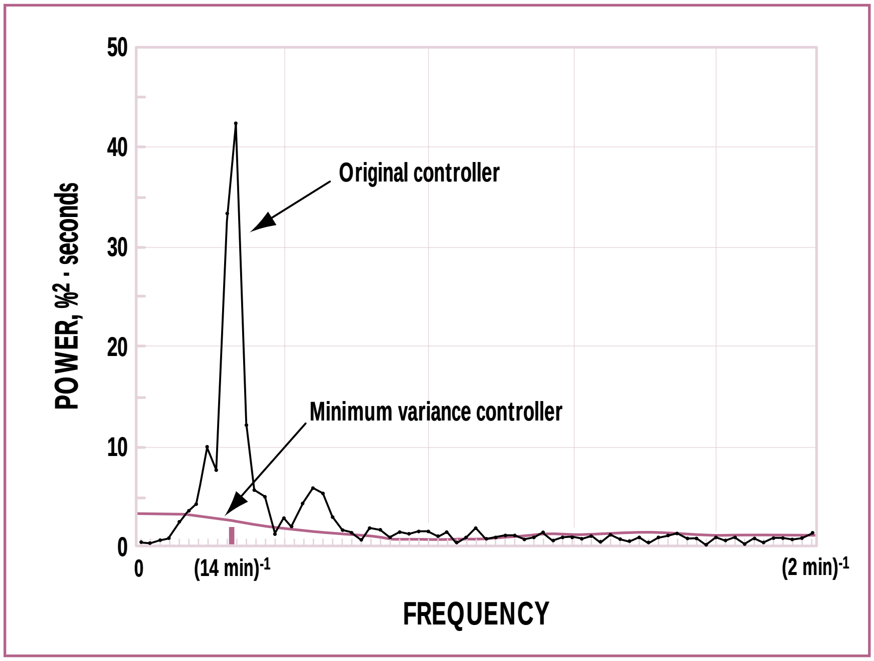
No feedback controller can react before it detects that the process output has deviated. The controller reacts by changing the process input. During the time it takes for the change to show up at the process output, the deviation persists. The product variance is a direct consequence of the lag, or process dead-time.

LINK TO STATISTICAL PROCESS CONTROL

The concept of minimum variance is recognized in statistical process control theory *(3)*. Random, "unassignable" causes produce the inherent process variability, which is unavoidable.

In statistical process control, the process is said to be "capable" only if the control limits are at least ±3 standard deviations from the mean. Intervention is unwarranted while the process remains within the control limits. That much variation is inherent in the process and cannot be reduced except by taking comprehensive measures such as physically changing the process, retooling, and retraining operators.

Even a perfect controller cannot reduce the variance below the minimum variance. If a control loop achieves variance near the minimum, it is pointless to tune the control or implement more sophisticated controls. Your best investment is in a loop in which the current variance is much larger than the minimum variance.



1. Power spectra for stock consistency

POWER SPECTRUM

The limitations of a feedback controller often appear as peaks in the power spectrum. A power spectrum *P(f)* describes the distribution of variance s^2_Y as a function of frequency *f*, where *Y* denotes the process output. The integral of the power spectrum equals the total variance:

See page 101 for equation 1

Sometimes a distribution has a significant peak or lobe at a particular frequency. If it does, then a disproportionate amount of the total variance is attributable to cycling at that frequency, and the variance is said to be "colored."

Figure 1 shows a power spectrum for stock consistency as measured in a paper mill. The original controller results in a strong peak with a cycle period of 14 min. Under minimum variance control, the spectrum peaks at low frequencies, which is a common result. The area under the minimum variance spectrum is substantially less than the area under the original spectrum. In other words, the total variance would be significantly less under minimum variance control than under the original controller.

As the figure shows, most of the variance reduction is achieved by eliminating the strong peak at a frequency of (14 min)$^{-1}$. However, greater variance occurs at the lowest frequencies, as shown on the far left side of the graph. The price of rejecting the mid-range variance is additional long-term-variation. A flat power spectrum is impossible.

Autocovariance function

The momentum of a process causes variations to endure or persist over time. This persistence is described by the autocovariance function. Equation 2 defines the power spectrum as the Fourier transform of the autocovariance function *(4)*. For $0 \leq f$,

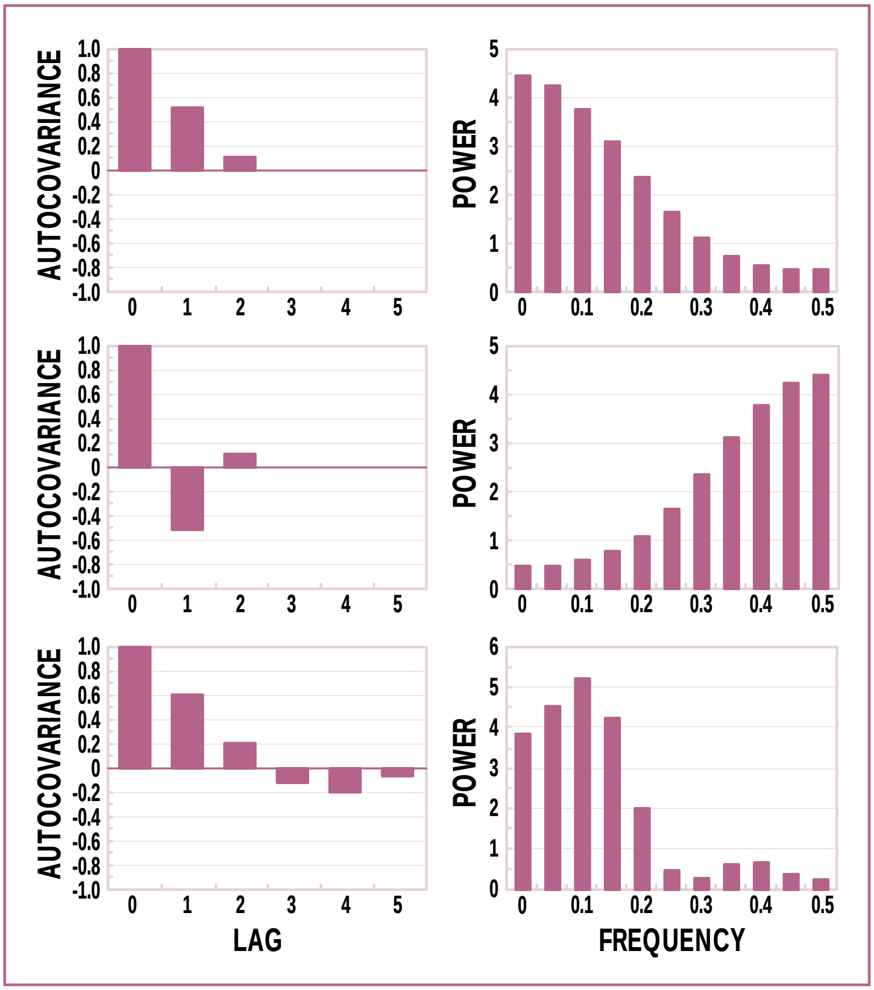
$$P(f) = 2\,F\,\{R(k)\} \qquad (2)$$

Here, *R(k)* is the autocovariance function at Lag *k*, and *F* stands for the Fourier transform.

Figure 2 illustrates that high-, medium-, and low-frequency lobes in the power spectrum are possible, depending on the nature of the autocovariance function. The graphs on the left show hypothetical autocovariance functions, and the graphs on the right show their Fourier transforms.

Most industrial processes have autocovariance functions like that depicted in the two graphs at the top of the figure. The process dead-time is two lags, and the process's autocovariance function has positive values at Lags 1 and 2. The power spectrum of the process displays a large lobe at low frequencies. This long-term variation is not a result of poor controller tuning but rather an unavaoidable consequence of the process itself. In this case, no controller can reduce the low-frequency variance.

The middle two graphs in Fig. 2 illustrate another example common in many industrial processes. The process dead-time is again two lags, but the negative autocovariance coefficient at Lag 1 reflects a high-frequency oscillation leading to short-term variation in product quality. A control engineer might be tempted to think that controller tuning could eliminate the short-term variation. However, the process dead-time makes this impossible. The process itself must be modified.



2. Hypothetical autocovariance functions (left) and their power spectra (right)

If the same process were sampled twice as fast, the result might be the autocovariance function shown in the lower graphs of the figure. The number of non-zero autocovariance coefficients doubles, and the peak in the power spectrum moves to an intermediate position. This outcome demonstrates that the interpretation of short-, medium-and long-term variation depends on the sampling period.

CONTROLLER PERFORMANCE INDEX

Lobed spectra make it difficult to assess controller performance. For example, Fig. 3 shows two spectra: one flat and one lobed. The flat spectrum reflects performance under a controller that results in "white noise" for the process variance. The absence of dominant cycling in the flat, "white noise" spectrum might be taken as an indication of good performance, but that could be a mistake.

Suppose the lobed spectrum is for the same process under an alternative controller. The alternative controller would be better because it eliminates more variance at high frequencies. With respect to the flat spectrum, the alternative spectrum does not have a peak at low frequencies; instead, it has a valley at high frequencies.

For benchmarking feedback controllers, Desborough and Harris proposed a performance index, [h]*(b, f)*, that uses minimum variance control as the benchmark *(5)*. At each frequency, the index is the difference between the observed power $P_Y(f)$ and the theoretical power under minimum variance control $P_{mv}(f)$. The index is normalized by dividing the difference by the total variance $[s]^2_Y$. (The appendix outlines how to estimate $P_{mv}(f)$ from closed-loop operating data.)

$$[h](b, f) = [P_Y(f) - P_{mv}(f)]/[s]^2_Y \qquad (3)$$

Harris showed how to calculate $P_{mv}(f)$ from closed-loop data without disturbing the process and without actually implementing a minimum variance controller *(2)*. The minimum variance power is independent of the controller.

Figure 4 shows the performance index for the stock consistency under the control loop having the power spectra shown in Fig. 1. The performance index compresses the information in the two power spectra into a single function. Where the index is 0, the controller is regulating the process to perform as effectively as it would under a minimum variance controller.

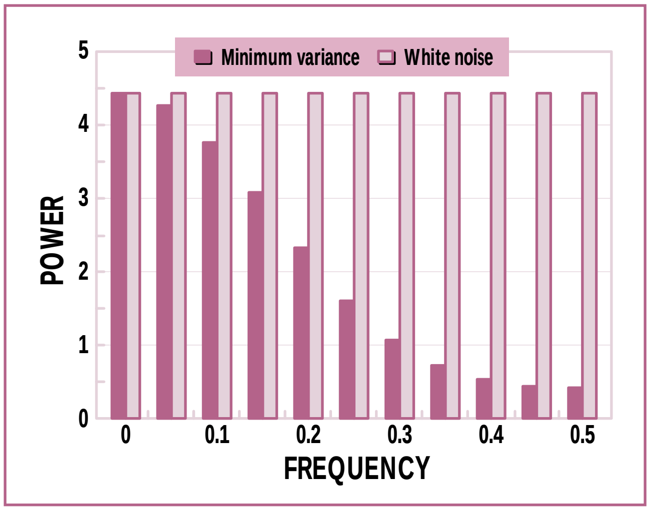
At a frequency of $(14\ \text{min})^{-1}$, the index has a value of 0.2. At this frequency, minimum variance control would reduce the variation substantially. At other frequencies, the index is close to the target value of 0.

For gross comparisons, the integrated index h*(b)* is convenient.

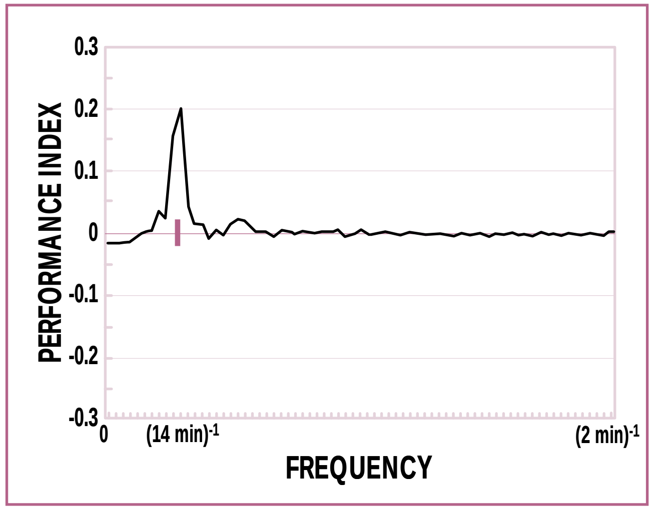
See page 101 for equation 4

where h^2_{mv} is the minimum variance and s^2_Y is the current variance.

For the stock consistency loop, the integrated performance index h*(b)* is 0.5.



3. Comparison of lobed and white noise power spectra



4. Performance index for stock consistency control

KEYWORDS

Automatic control, controllers, control systems, feedback control, investment, models, monitoring, parameters, performance evaluation, process control, variance.

$$\eta(b) = \int_{f=0}^{\infty} \eta(b,f)\,df$$

$$= \frac{\sigma_Y^2 - \sigma_{mv}^2}{\sigma_Y^2}$$

$$= 0.5$$

This indicates that the original controller results in twice the total variance of a minimum variance controller.

The performance index is applicable to all linear, one-variable feedback controllers, including proportional–integral controllers, Smith predictors, and internal model controllers. The notation h*(b)* emphasizes that the inherent minimum variance is attributable to the dead-time *b*.

CONCLUSIONS

Diagnosing control problems and identifying opportunities for reducing variance requires a benchmark that reflects the limitations of the process. A suitable baseline is especially helpful if the process has substantial dead-time, because much of its variability will be inherent and unavoidable. Minimum variance control is an appropriate benchmark for *achievable* performance.

Control loops that deliver product at variances well above the minimum are good candidates for maintenance. In these loops, changes to the controller will result in substantial improvements. Conversely, loops that are running close to minimum variance are poor candidates for controller maintenance, because no controller can reduce the variance below the minimum.

All the control loops are competing for your limited maintenance time and money. The performance index levels the playing field and highlights your best investment opportunities. TJ

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LITERATURE CITED

1. Stanfelj, N., Marlin, T. E., and MacGregor, J. F., *American Control Conference Proceedings*, American Automatic Control Council, 1991, pp. 2886–2892.
2. Harris, T. J., *Can. J. of Chem. Eng.* 67(Oct): 856(1989).
3. Berger, R. W. and Hart, T. H., *Statistical Process Control: A Guide for Implementation*, Am. Soc. for Quality Control, 1986, p. 23.
4. Bendat, J. S. and Piersol, A. G., *Engineering Applications of Correlation and Spectral Analysis*, Wiley, New York, 1993.
5. Desborough, L. and Harris, T., *Can. J. Chem. Eng.* 70(Dec.): 1186(1992).

EQUATIONS

$$\int_0^{\infty} P(f)\,df = \sigma_Y^2 \qquad (1)$$

$$P(f) = 2\ \mathrm{F}\ [R(k)] \qquad (2)$$

$$\eta(b) = \int_{f=0}^{\infty} \eta(b, f)\,df = \frac{\sigma_Y^2 - \sigma_{mv}^2}{\sigma_Y^2} \qquad (4)$$

$$Y_i = \sum_{i=0}^{\infty} \psi_i a_{t-i}$$
$$\psi_0 = 1 \qquad (5)$$

$$\sigma_{mv}^2 = \sigma_a^2 \sum_{i=0}^{b-1} \psi_i^2 \qquad (6)$$

For $\| k \| < b$,

$$R_{mv,k} = \sigma_a^2 \sum_{i=0}^{b-1-\|k\|} \psi_i \psi_{i-\|k\|} \qquad (7)$$

For $\| k \| \geq b$,

$$R_{mv,k} = 0 \qquad (8)$$

APPENDIX: ESTIMATING THE MINIMUM VARIANCE POWER SPECTRUM

To calculate the minimum variance power spectrum, we begin by modeling the process. A moving average model is convenient.

See page 101 for equation 5

Here, Y_t is in the process output sampled at the discrete time t. For simplicity, Y_t is defined as the deviation from setpoint, so it has a mean of zero. The y_i are model parameters, and a_t is a white noise signal with a variance s^2_a.

If we have the model parameters, Eqs. 6–8 allow us to estimate the minimum variance s^2_{mv} and its autocovariance $R_{mv,k}$.

See page 101 for equation 6

See page 101 for equation 7

See page 101 for equation 8

For a process with dead-time b, only s^2_a and $Y_0 \ldots Y_{b,i}$ are needed. Harris showed that these parameters are independent of the controller *(5)*. Therefore, it is possible to estimate the model parameters even from closed-loop data.

The minimum variance power spectrum is calculated as the Fourier transform of the minimum variance autocovariance function. The autocovariance function is symmetric, so the Fourier cosine transform applies.