The effect on capability indices of small shifts in the process level.

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ABSTRACT

An industrial process may well be centered on the target value, but more or less permanent, small deviations are often unavoidable. They are usually due to the following factors, which can be classified as: (a) a permanent deviation from the target value (a systematic bias); (b) assignable causes (shift-to-shift effects, machine-to-machine differences, etc.); and (c) common causes (usually random noise). Assignable causes could be identified and removed. On the other hand, common causes (mainly random noise) are hard or almost impossible to identify and to remove. In quality assurance, steady state, or equivalently, the absence of assignable causes was earlier a necessary requirement for using capability indices in a meaningful way. In this study, the confidence intervals for a capability index are derived, allowing the precence of the above mentioned factors under the assumption that the assignable causes can be modeled statistically. Further, the effects of removing these factors are also practically demonstrated. Particular attention has been made to the Taguchi capability index, C_{pm}, since it depends both on the variability of the process and on the deviation of the process mean from the target value.

Application:

A practical application of the results given in the article is where there are small but distinguishable shifts in the process level and when there are needs to reduce these shifts to a minimum.

CAPABILITY INDICES HAVE BEEN widely used during the last 10 years. Some of the reasons for this popularity are: Most manufacturing companies have become aware of the importance of variability. The indices are easy to compute and a single number summary considerably simplifies the supervision of a production process. Also, the indices will provide appropriate information about alternative processes or suppliers, performance problems, etc. However, capability indices have been used uncritically because of their popularity. Some authors have severely criticized this practice; for example, to quote Gunter, "the greatest abuse of C_{pk} that I have seen is that it becomes a mindless effort that managers confuse with real statistical process control efforts" *(1)*.

CAPABILITY INDICES

A brief review of some different indices are given below:

C_p is defined as:

$$C_p = (U - L)/6s \qquad (1)$$

where U is the upper specification level for the process, L the lower specification level and s the standard deviation of the population. This index only provides information about the variability of the process in relation to its specification limits.

The most common capability index is

$$C_{pk} = \min\,[(\mu - L)/3s,\ (U - \mu)/3s] \qquad (2)$$

where µ is the population, process mean. The C_{pk} includes both the location of the process level and the variability of the process and many customers require knowledge about the C_{pk} for a lot.

However, if both the variability, the location of the process level, and the deviation from the target value, T, is important, then the capability index C_{pm} is recommended. C_{pm} is also referred to as the Taguchi capability index and is defined as:

$$C_{pm} = (U - L)/\{6\mathrm{SQRT}[s^2 + (\mu - T)^2]\} \qquad (3)$$

CHOICE BETWEEN DIFFERENT CAPABILITY INDICES

In general, the parameters µ and s, and hence the indices, are unknown, which means that estimates of them have to be used, e.g., $\bar{X}$ for the process mean, µ and the standard deviation, S, for s while $\hat{C}_p$, $\hat{C}_{pk}$, and $\hat{C}_{pm}$ are used as es mates for the indices.

The relation between the standard deviation and $\bar{X}$ for $\hat{C}_p = \hat{C}_{pk} = \hat{C}_{pm} = 1.33$ is shown in Fig. 1, which is based on a time series of observations from an arbitrary, industrial process as given in Fig. 2.

It can be seen from Fig. 1 that the value of $\hat{C}p$ is equal or larger than1.33 as long as $\bar{X}$ stays inside the interval between the upper specification limit (USL) and the lower specification limit (LSL) and as long as the standard deviation remains equal or less than 0.62. $\hat{C}$pkremains equal or larger than 1.33 even if $\bar{X}$ approaches any of the specification limits, USL or LSL when the standard deviation approaches zero. $\hat{C}p$ and $\hat{C}pk$ are sometimes used simultan ously to avoid this pitfall. $\hat{C}_{pm}$ is equal to or larger than 1.33 for a closer range of values on both the standard deviation and $\bar{X}$. The feasible region for these values are within the boundaries given by $\hat{C}_{pm} = 1.33$ in the standard deviation – $\bar{X}$ diagram.

PROCESS STABILITY

Both Deming *(2)* and Shewhart *(3)* assume that the quality characteristic is independently, normally distributed and that the process is stable. The main reason for the last assumption is due to the fact that the process should be predictable; otherwise, the capability index will have no meaningful interpretation. Deming also requires that "Quality characteristics remain nearly constant hour after hour, day after day." He comes to the conclusion that capability is definable if and only if:

1. The quality characteristic can be properly described by some distribution.
2. Both the process mean and the process variance are constant in time.

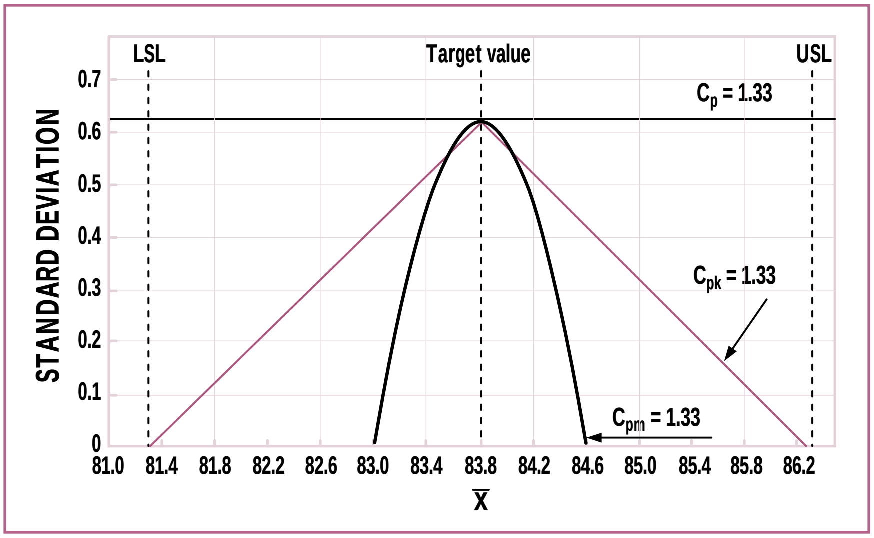
Shewhart, on the other hand, is not that restrictive. He just requires that "A phenomenon will be said to be controlled when, through the use of past experience, we can predict, at least within limits, how the phenomenon may be expected to vary in the future," which corresponds to condition 1 above. This implies that if it is possible to find a suitable statistical model describing the quality characteristic, the capability is definable and can be interpreted. Shewhart's weaker definition allows capability studies on a larger set of production processes.

In this paper, the study is generalized to the common situation where an industrial process may be well centered at the intended process mean (process level) on or close to the target value but cannot be steadily kept at that level. Deviations of the process level from the target value are due to more or less distinguishable factors, assignable causes, e.g., shift-to-shift, machine-to-machine differences or variations in input raw material, etc. When such factors exist, a traditional interpretation of the indices can be misleading. However, if these factors can be modeled statistically, then, in line with the thinking of Shewhart, the capability indices can be calculated and interpreted.

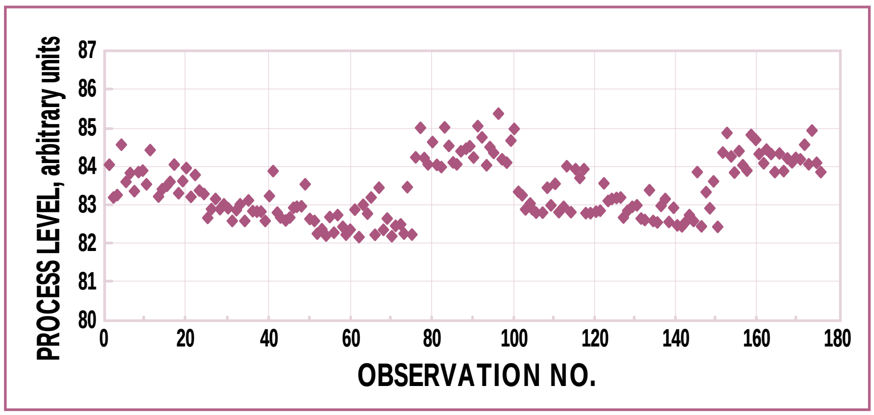
In the present study, the instability of the production process is restricted to permanent and/or occasional shifts in the process level. We define occasional shifts as the subgroup of assignable causes, which can be statistically modeled; and, in this study, we use a one-way analysis of variance (ANOVA) model. A discussion of the statistical model is given below.

Please also refer to Fig. 3, which is the same as Fig. 2, but with the estimated shifts in the process level marked with a dotted line.

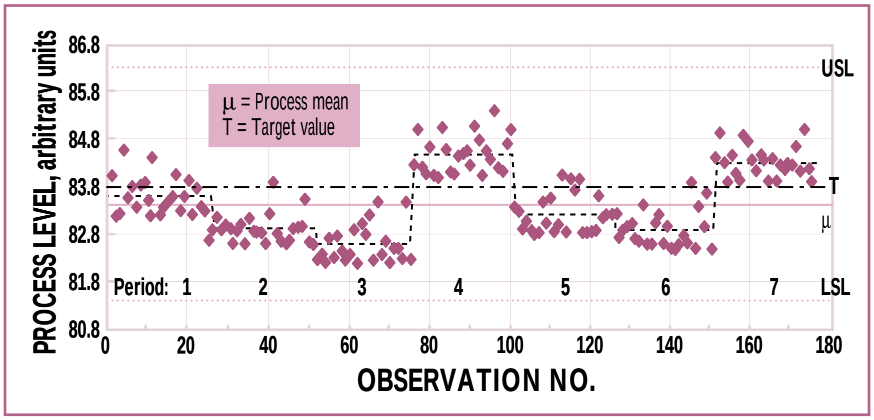
An obvious question is: How does an elimination, or a decrease in size, of the occasional factors affect the value of $\hat{C}_{pm}$ and its corresponding confidence interval? In order to answer such a question, approximate lower confidence intervals for $\hat{C}_{pm}$ must first be derived. It will then be possible to study the effects of eliminating and/or decreasing different factors.



1. The relation between the standard deviation, S, (std, dev) and the measured process mean ($\bar{X}$) for $\hat{C}_{pk} = 1.33$, $\hat{C}_{pk} = 1.33$ and $\hat{C}_{pm} = 1.33$. The upper and lower specification limits and target value according to Fig. 2.



2. Time series with 175 observations from a lot manufactured during a period of 7 days.



3. Time series with 175 observations as in Fig. 2. Dotted line = shifts in the process level.

STATISTICAL MODEL

Let X_i denote the ith measurement of a quality characteristic of a manufacturing process. The mean value of the process is μ (the process level), the variance s^2 and the target value of the process, T.

In general, it is assumed that the quality characteristic varies around the mean so that X_i can be written:

$$X_i = \mu + ei = 1, 2, \ldots, t \qquad (4)$$

where t is the number of observations and e_i is the noise factor. The latter is commonly assumed to be normally distributed with the mean 0 and the variance s^2. The parameter μ is estimated by $\mu = \bar{X}$ and the variance with $\hat{\sigma}^2$ where:

$$\hat{\sigma}^2 = (t-1)^{-1}\sum_{i=1}^{t}\left(X_i - \bar{X}\right)^2 \qquad (5)$$

which, for instance, is used as the estimator of the variance in the denominator of the $\hat{C}_p$ index.

Let us make the same assumptions as above, but assume instead that we are interested in the variability around the target value, T.

Then the process can be modeled as:

$$(X_i - T) = (\mu - T) + e_i, \; i = 1, 2, \ldots, t \qquad (6)$$

The estimator of the variation of the process around T, $\hat{MSE}$, (mean square error), becomes:

$$\hat{MSE} = t^{-1}\sum_{i=1}^{t}(X_i - T)^2 \qquad (7)$$

The C_{pm} index can be written as $C_{pm} = (U - L)/6(MSE)^{1/2}$, where $MSE = s^2 + (\mu - T)^2$ and the estimator, $\hat{C}_{pm}$ becomes:

$$\hat{C}_{pm} = (U - L)/[6(\hat{MSE})^{1/2}] \qquad (8)$$

It is assumed in this paper that the process mean, μ, randomly varies in time, e.g., $\mu = \mu_i$. Since we are interested in the variability of the process around the target value, T, the statistical model of the process becomes :

$$X_{ij} - T = \mu_i - T + e_{ij} \qquad (9)$$

:where

$i = 1, 2, \ldots, t$ (number of periods)

$j = 1, 2, \ldots, r$ (number of observations per period).

A generalized model of the process with an arbitrary number of observations per period would be complex to analyze, although the techniques given below could be used for such a purpose. For reasons of simplicity, it has been deliberately chosen in this paper to simplify the process model by assuming the same number of observations per period, r. With this assumption, the local mean can be written as $\mu_i = \mu + a_i$, where $\sum_{i=1}^{t} a_i = 0$. The process can now be written as:

$$X_{ij} - T = \mu + a_i - T + e_{ij} \qquad (10)$$

MSE and its estimator $\hat{MSE}$ become, respectively, Eqs. 11 and 12:

$$MSE = s^2 + t^{-1}\sum_{i=1}^{t} a_i^2 + (\mu - T)^2 \qquad (11)$$

$$\hat{MSE} = (tr)^{-1}\sum_{i=1}^{t}\sum_{j=1}^{r}(X_{ij} - T)^2 \qquad (12)$$

It can be shown that the expected value of $\hat{MSE}$, $E(\hat{MSE})$ is equal to MSE, i.e.:

$$E(\hat{MSE}) = MSE = s^2 + t^{-1}\sum a_i^2 + (\mu - T)^2 \qquad (13)$$

The square of the ratio between the capability index of the process, C_{pm} and its estimator, $\hat{C}_{pm}$ can then be written:

$$[C_{pm}/\hat{C}_{pm}]^2 = \hat{MSE}/MSE = (tr)^{-1}\sum_{i=1}^{t}\sum_{j=1}^{r}(X_{ij} - T)^2/[s^2 + (1/t)\sum a_i^2 + (\mu - T)^2] \qquad (14)$$

Further, it can be shown that the ratio $[C_{pm}/\hat{C}_{pm}]^2$ approximately follows a chi-square distribution divided by its degrees of freedom, v, d.f., then:

$$[C_{pm}/\hat{C}_{pm}]^2 \approx c^2(\hat{v})/\hat{v} \qquad (15)$$

Details are given in the Appendix.

When the distribution is known, confidence intervals can then be constructed in the following way:

Let $c^2_{1-a}(n)$ denote the tail percentile points of the chi-square distribution and z_a the corresponding values for the normal distribution.

An approximate, 100 (1 – a) %, lower confidence limit based on Eq. 15 becomes:

$$\hat{C}_{pm}SQRT[c^2_{1-a}(\hat{v})/\hat{v}] \qquad (16)$$

For large samples, we use the fact that $SQRT[2c^2(\hat{v})]$ is approximately normally distributed and the corresponding lower confidence limit is:

$$\hat{C}_{pm}[1 - z_a/(2\hat{v})^{1/2}] \qquad (17)$$

Grand $\bar{X}$	Grand std. dev	Grand $\hat{C}_p$	$\hat{C}_{pk}$	$\hat{C}_{pm}$
83.4	0.75	1.11	0.93	0.95

I. Grand $\bar{X}$, standard deviation, $\hat{C}_p$, $\hat{C}_{pk}$ and $\hat{C}_{pm}$ for the time series in Fig. 3.

Period	X	Std. dev.	$\hat{C}p$	$\hat{C}pk$	$\hat{C}pm$
1	83,5	0,33	2,49	2,19	1,85
2	82,9	0,30	2,80	1,79	0,88
3	82,6	0,33	2,49	1,28	0,66
4	84,4	0,28	3,03	2,28	1,23
5	83,1	0,33	2,50	1,80	1,07
6	82,8	0,26	3,15	1,89	0,81
7	84,3	0,38	2,20	1,78	1,35

II. Periodical means, standard deviations, $\hat{C}_p$, $\hat{C}_{pk}$, and $\hat{C}_{pm}$ for the time series in Fig. 3.

Percentiles for the chi-square- and the normal distribution can be found in ordinary statistical tables, and Boyles *(4)* has computed lower confidence limits as a function of n.

SHIFTS IN THE PROCESS LEVEL

First, let us consider all the 175 observations as taken from one period and calculate the mean, standard deviation, and all the capability indices, which are given in Table I.

Now, let us focus our attention on the effects of shifts in our data as indicated in Fig. 2 and Fig. 3. The corresponding, calculated, values for the periods are given in Table II.

As shown in Table I, Table II and Fig. 4, it can be seen that a shipment from a lot manufactured during periods 2, 3, 5 and 6, would never have been accepted under the condition $\hat{C}_{pm} \geq 1.33$, while it most likely had been accepted if the quality assurance would have been done under the condition $\hat{C}_{pk} \geq 1.33$. Also, as indicated in Table II, a large value of $\hat{C}_p$ won't tell us anything about the deviation of the process level from the target value. It is also graphically well indicated in Fig. 1 that a fixed value on $\hat{C}_{pm}$ will result in an upper limit for the absolute difference $|\bar{X} - T|$ as the standard deviation approaches zero.

Finally, it can be shown that the value of $[\hat{C}_{pm}]^2$ is related to the periodical $[\hat{C}_{pm}]^2$ – values through the relation:

$$1/[\hat{C}_{pm}]^2 = t^1 \sum_{i=1}^{t} 1/[\hat{C}_{pm\ i}]^2$$

@text:Such a relation does not exist for the other indices.

Source	Sum of squares (SS)	d.f.	MSE
Permanent factor	30.87	1	30.87
Temporary factor	78.87	6	13.145
Noise factor	24.22	168	0.14417
Total		133.96	175

III. ANOVA table according to Table V in the Appendix and with data computed from the observations as referred to by the time series in Fig. 3.

Eliminated components confidence interval	$\hat{C}pm$	$\hat{C}pm$ Lower 95%
1. None.	0.95	0.90
2. Permanent factor, T=μ; i.e., process level on the target value	1.14	1.07
3. Temporary factor, a i = 0; i = 1,2..7; i.e., no shift effects	1.48	1.37
4. Both permanent and temporary factors, i.e., process level on target value and no shift effects	2.24	2.04

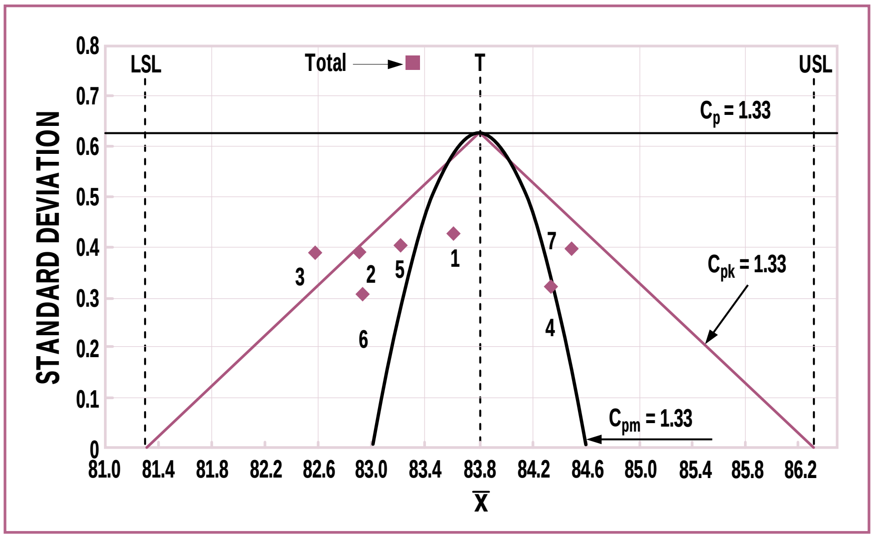
*IV. The estimated Taguchi index, *sym* and its lower confidence interval under different conditions from the observations as referred to by the time series in Fig. 3.*

KEYWORDS

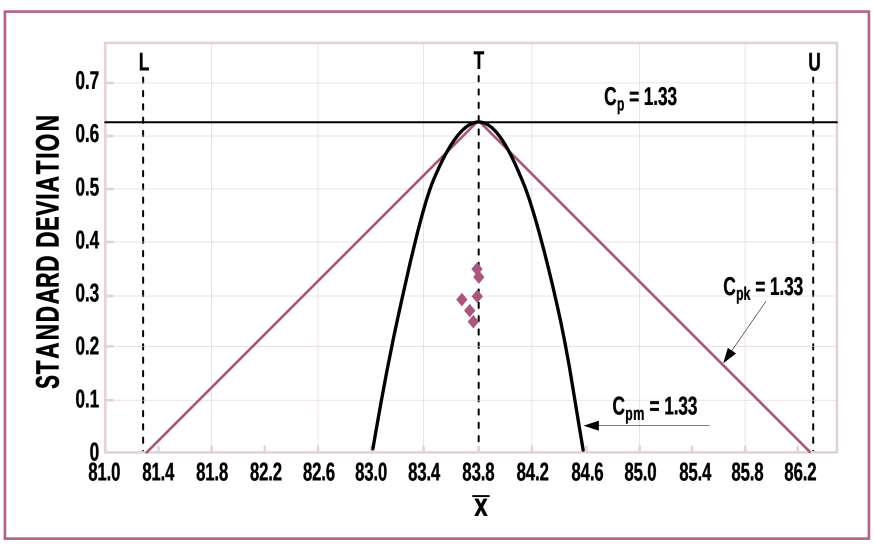
Capacity, mathematical models, process variables, random error, stability, statistical analysis, statistical process control.

Source	Sum of squares	d.f.	MSE
Permanent factor	$SS_{PERMANENT} = tr(X\ \ - T)^2$	1	$MS_{PERMANENT}$; $s^2 + tr(\mu - T)^2$
Temporary factor	$SS_{TEMPORARY} = r\Sigma(X_j\ X.)^2$	(t-1)	$MS_{TEMPORARY}$; s^2 + r(t-1)- 1∑i=1t[alpha]i2
Noise factor (error)	$SS_{NOISE} = \Sigma\Sigma(X_{ij} - X_j)^2$	t(r-1)	MS_{NOISE};s^2
Total	$SS_{TOT} = \Sigma\Sigma(X_{ij} - T)^2$		

V. A formal ANOVA-table with sum of squares, degrees of freedom and expected mean square error for the permanent factor, temporary (or shift) factor and the noise (error) factor.



4. Plot of process mean and standard deviation for the various shifts as indicated by Fig. 3 and for the grand total compared with $\hat{C}_{pm}$ = 1.33.



5. Plot of $\hat{C}_{pm}$ = 1.33 in a standard devation (std. dev.) and process mean ($\bar{X}$)graph, where both the permanent and the temporary factors have been eliminated from the data as originally given in Fig. 2.

PRACTICAL CONSEQUENCES OF SMALL SHIFTS IN THE PROCESS LEVEL

As shown by Eq. A7 in the Appendix, the total variability of the process can be decomposed into partial sums. An obvious question then is: "How does an elimination of one or more factors affect the value of $\hat{C}_{pm}$ and its corresponding confidence interval?"

Analysis of variance, ANOVA, allows us to separate the total variability around T in three different components:

1. Permanent factor, $\mu - T$
2. Temporary factor, $a_i = \mu_i - \mu$
3. Noise factor, e_{ij}.

and it will be possible to study the effect of each one of the factors on the $\hat{C}_{pm}$ index. A one-way ANOVA table of the data becomes Table III.

By inserting our data in Eqs. A7 and A8 in the Appendix and by using the data given in Table III, we obtain the results given in Table IV. It can be seen from Table IV that the increase in $\hat{C}$pm is not additive. In this particular case, each elimination of a factor increases the value of $\hat{C}$pm with 0.19 and 0.53 units, respectively. If both the permanent and the temporary factors are eliminated, the value of $\hat{C}_{pm}$ increases from 0.95 to 2.24.

If, from a practical point of view, it would only be possible to eliminate one factor, in this case, priority should be given to the shift effects. This is in line with modern quality thinking—rather a homogenous process at a wrong level than a heterogenous one at the right level, which is also well indicated by Figs. 5 and 6. TJ

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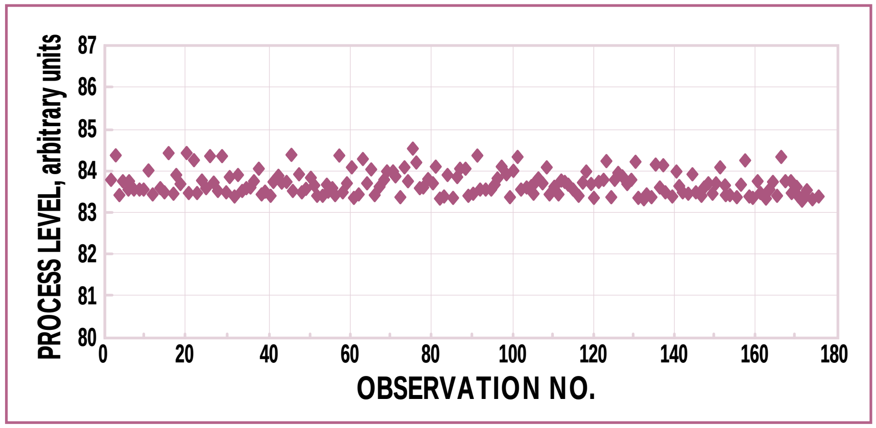

6. Time series with 175 observations where both the permanent and the temporary factors have been eliminated from the data as originally given in Fig. 2.

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APPENDIX

Statistical model

Assume that:

$$X_{ij} = \mu + a_i + e_{ij} \quad \text{(A1)}$$

where

i= 1, 2, . . . ,*t* (number of periods)

j= 1, 2, . . . ,*r* (number of observations per periods)

where $e_{ij} = N(0, s^2)$ and the constraint $\sum a_i = 0$.

Further, let T be the target value and denote $\widehat{MSE}$ as $\hat{\tau}^2$:

$$\hat{\tau}^2 = (tr)^{-1}\sum\sum(X_{ij} - T)^2 \quad \text{(A2)}$$

Carlsson *(5)* has shown that:

$$tr\,\hat{\tau}^2/s^2 = \sum\sum(X_{ij} - T)^2/s^2 \quad \text{(A3)}$$

follows a non-central chi-square distribution, i.e.:

$$tr\,t^2/s^2 = c^2(tr, trQ^2 \quad \text{(A4)}$$

with *tr* degrees of freedom and with the non-centrality parameter trQ^2, where:

$$Q^2 = (1/t)\sum a_i^2 + (\mu - T)^2/s^2 \quad \text{(A5)}$$

Further, a non-central c^2-distributed random variable is approximately c^2-distributed but with an other number of degrees of freedom, n.

Boyles *(4)* and Carlsson *(5)* have shown that $[C_{pm}/\hat{C}_{pm}]^2$ is approximately distributed as $c^2(n)/n$. The number of degrees of freedom can be estimated by:

$$\hat{v} = tr(1 + \hat{\theta}^2)/(1 + 2\hat{\theta}^2) \quad \text{(A6)}$$

For this particular statistical model, Carlsson *(5)* has shown that the estimator of the non-centrality parameter in the approximate distribution, $tr\,\hat{\theta}^2$, can, (see Table V), be calculated as:

$$tr\,\hat{\theta}^2 = [(t - 1)MS_{TEMP}/MS_{NOISE} + MS_{PERM}/MS_{NOISE}] - t \quad \text{(A7)}$$

The total sum of squares, SS_{TOT}, given in Table V, can be separated into a number of independent sums of squares, i.e.:

$$SS_{TOT} = SS_{PERMANENT} + SS_{TEMPORARY} + SS_{NOISE} \quad \text{(A8)}$$

and the mean sum of squares: $MS_{PERMANENT}$, $MS_{TEMPORARY}$ and MS_{NOISE} can be calculated from the ANOVA table (Table V). by dividing the values of the sum of squares with the corresponding degrees of freedom, d.f.